This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Box, No.

CANCELLED

ESSEX INSTITUTE.

PRESENTED BY

The Library Committee shall divide the books and other articles belonging to the Library into three classes, namely: (a) those which are not to be removed from the building; (b) those which may be taken only by written permission of three members of the committee; (c) those which may circulate under the following rules:—

Members shall be entitled to take from the Library two folio or quarto volumes, or four volumes of lesser fold, with the plates belonging to the same, upon having them recorded by the Librarian, or Assistant Librarian, and promising to make good any damage they sustain, while in their possession, and to replace the same if lost, or pay a sum fixed by the Library Committee.

No person shall lend any book belonging to the Institute, excepting to a member, under a penalty of one dollar for each offence.

The Library Committee may allow members to take more than the allotted number of books upon a written application, and may also permit other persons than members to use the Library under such conditions as they may impose.

No person shall detain any book longer than four weeks from the Library, if notified that the same is wanted by another member, under a penalty of five cents per day, and no volume shall be detained longer than three months at one time under the same penalty.

The Librarian shall have power by order of the Library Committee to call in any volume after it has been retained by a member for ten days.

On or before the first Wednesday in May, all books shall be returned to the Library, and a penalty of five cents per day shall be imposed for each volume detained.

No book shall be allowed to circulate until one month after its receipt.

HARVARD COLLEGE LIBRARY

THE GIFT OF

GEORGE ARTHUR PLIMPTON

3 2044 097 010 805

NEW

ELEMENTARY ALGEBRA:

IN WHICH

THE FIRST PRINCIPLES OF ANALYSIS

ARE PROGRESSIVELY DEVELOPED AND SIMPLIFIED.

FOR

COMMON SCHOOLS AND ACADEMIES.

BY BENJAMIN GREENLEAF, A. M.,
AUTHOR OF A MATHEMATICAL SERIES.

SIXTEENTH ELECTROTYPE EDITION.

BOSTON:
PUBLISHED BY ROBERT S. DAVIS & CO.
NEW YORK: WILLIAM WOOD AND COMPANY, AND W. I. POOLEY.
PHILADELPHIA: J. B. LIPPINCOTT AND COMPANY
ST. LOUIS: KEITH AND WOODS.
NEW ORLEANS: DUNCAN & CO.
1867.

Educ T 128.67.420
✓

HARVARD COLLEGE LIBRARY
GIFT OF
GEORGE ARTHUR [illegible]
JANUARY 25, 1924

GREENLEAF'S

MATHEMATICAL SERIES.

I. **Greenleaf's New Primary Arithmetic**; an attractive book of easy lessons for beginners. 84 pp.

II. **Greenleaf's New Intellectual Arithmetic**; a *late* work, with models of analysis. 180 pp.

III. **Greenleaf's Common School Arithmetic**; a complete system of Written Arithmetic for Common Schools. 324 pp.

IV. **Greenleaf's National Arithmetic**; a thorough course for High Schools, Academies, Normal Schools, and Commercial Colleges. 444 pp.

V. **Greenleaf's New Elementary Algebra**; or the First Principles of Analysis for Schools and Academies. 324 pp.

VI. **Greenleaf's New Higher Algebra**; an advanced Analytical Course, for High Schools and Colleges. 394 pp.

VII. **Greenleaf's Elements of Geometry**; with applications to Mensuration. 320 pp.

VIII. **Greenleaf's Elements of Trigonometry**; with Practical Applications and Tables. 170 pp.

IX. **Greenleaf's Geometry and Trigonometry**; or the last two works in one volume. 490 pp.

X. **Greenleaf's Surveying and Navigation**, with Practical Applications and Tables. [*In preparation.*]

☞ Keys to the Arithmetics, Algebras, Geometry and Trigonometry. For Teachers only. 6 volumes.

Entered according to Act of Congress, in the year 1862, by
BENJAMIN GREENLEAF,
in the Clerk's Office of the District Court of the District of Massachusetts.

CAMBRIDGE: PRINTED BY H. O. HOUGHTON AND COMPANY.

PREFACE.

TEN years ago the author presented to the public his Treatise on Algebra, a comprehensive theoretical and practical work. The generous favor with which that book has been greeted, some forty editions having been sold, indicates that he in some degree provided for the wants of classes in that department.

But it has been noticed that, in consequence of the improved condition of our public schools at the present time, pupils are enabled to complete their arithmetical studies at a comparatively early age; and in consequence, a demand has arisen for an easy algebraic course to follow. To meet this growing demand, this work has been prepared.

While the aim has been to render the course easy and simple, great care has been taken that this should not be secured at the expense of strength and thoroughness.

The analytic method has been pursued, with a view to a strictly logical development of the science, it being believed that one of the principal benefits of the study of mathematics is to teach the learner how to reason with elegance and exactness.

Brief articles on the Discussion of Problems, Rationalization, Radical Equations, and the Theory of Quadratic Equations, have been inserted to render the course more complete and better suited to the range of classes in academies as well as in common schools.

Every effort has been made to include valuable improvements, and all that is required by the best standards of instruction. To this end, the latest foreign works on the subject have been examined and compared, and the most prominent practical teachers of this country freely consulted.

Especial credit is here due to H. B. MAGLATHLIN, A. M., who has been associated with the author in the preparation of this book. To his correct scholarship, discriminating judgment, and ability as a mathematician, the value of this treatise is chiefly due.

BRADFORD, Mass., Sept. 1, 1862.

NOTE TO TEACHERS.

IN general, pupils who may have mastered the author's Common School Arithmetic, or any other similar book, are prepared to enter upon the study of this elementary course of Algebra.

For the convenience of those who require a less extended course, several articles have been marked by a *, to be omitted at the option of the teacher. Others, if deemed necessary, can be passed over without marring the unity of the work.

CONTENTS.

DEFINITIONS AND NOTATION.

PAGE
Signs 7
Algebraic Notation 10
Algebraic Expressions 13
Interpretation 16
Axioms 18
Algebraic Processes 19

ENTIRE QUANTITIES.

Addition 23
Subtraction 30
Multiplication 36
Division 45
Theorems 55
Factoring 59
Greatest Common Divisor . . . 66
Least Common Multiple 73

FRACTIONS.

Definitions 76
General Principles of Fractions . 77
Signs of Fractions 78
Reduction 80
Addition 89
Subtraction 92
Multiplication 94
Division 99

SIMPLE EQUATIONS.

Transformation of Equations . . 107
Solution of Simple Equations containing one Unknown Quantity 111
Problems leading to Simple Equations containing one Unknown Quantity 119
Simple Equations containing two Unknown Quantities . . . 137
Elimination 138
Problems leading to Simple Equations containing two Unknown Quantities 146
Simple Equations containing three or more Unknown Quantities . 153
Problems leading to Simple Equations containing three or more Unknown Quantities . . . 157

DISCUSSION OF PROBLEMS.

Interpretation of Negative Results . 161
Problems 163
Zero and Infinity 164
Problems 166

INVOLUTION.

Powers of Monomials 168
Powers of Fractions 170
Powers of Binomials 172
The Binomial Theorem 172

EVOLUTION.

Definitions 180
Square Root of Numbers 181
Cube Root of Numbers 186
Roots of Monomials 191
Square Root of Polynomials . . 193
Cube Root of Polynomials . . . 196

RADICALS.

Reduction of Radicals 201
Addition of Radicals 206
Subtraction of Radicals 208
Multiplication of Radicals . . . 209
Division of Radicals 213
Involution of Radicals 215
Evolution of Radicals 216
Rationalization 218
Imaginary Quantities 222
Radical Equations 224

QUADRATIC EQUATIONS.

Pure Quadratic Equations . . . 227
Simultaneous Equations 232
Problems leading to Pure Equations 234
Affected Quadratic Equations . . 236
First Method of Completing the Square 236
Second Method of Completing the Square 243
Third Method of Completing the Square 248
Equations in the Quadratic Form . 252
Simultaneous Equations 259
Theory of Quadratic Equations . 271
Formation of Equations 273
Discussion of the General Equation 274
Problems 276

RATIO AND PROPORTION.

Theorems relating to Proportion . 288
Problems in Proportion 293

SERIES.

Arithmetical Progression . . . 295
Problems 302
Geometrical Progression 304
Problems 311

Miscellaneous Examples 313

ELEMENTARY ALGEBRA.

DEFINITIONS AND NOTATION.

1. **Quantity** is anything that can be measured; as distance, time, weight, and number.

2. The **Unit** of quantity is one of the same kind as the quantity, taken as the standard, or unit of measure.

3. **Mathematics** is the science of quantities and their relations.

4. **Algebra** is that branch of mathematics in which quantities are represented by letters or other symbols, and their relations are indicated by signs. It has been called Universal Arithmetic.

SIGNS.

5. **Addition** is indicated by an erect cross, $+$, called *plus*. Thus, $10 + 4$, read ten plus four, signifies that 10 and 4 are to be added.

6. **Subtraction** is indicated by a short horizontal line, $-$, called *minus*. Thus, $10 - 4$, read ten minus four, signifies that 4 is to be subtracted from 10.

Define Quantity. Unity of Quantity. Mathematics. Algebra. How is Addition indicated? Subtraction?

7. MULTIPLICATION is indicated by an inclined cross, $\times$. Thus, 8×2 signifies that 8 and 2 are to be multiplied together.

In Algebra, the inclined cross is usually omitted, except between two arithmetical figures, separated by no other sign, and the absence of any sign indicates multiplication. Thus, $a\,b$ signifies that a and b are to be multiplied together.

NOTE. Sometimes a period is used in place of the inclined cross; but this should never be done when there is danger of mistaking it for the decimal point. Thus, $a\,.\,b$ signifies that a and b are to be multiplied together, and $2\,.\,3\,.\,5\,.\,7$ indicates that 2, 3, 5, and 7 are to be multiplied together; but $2\,.\,3$ would be read *two and three tenths*, unless the connection made it obvious that multiplication was intended.

8. DIVISION is indicated by a horizontal line, with one dot above and another below, $\div$. Thus, $8 \div 2$ signifies that 8 is to be divided by 2.

Division is otherwise often indicated by writing the dividend above and the divisor below a horizontal line, in the form of a fraction. Thus, $\frac{8}{2}$ signifies the same as $8 \div 2$.

NOTE. In expressing the ratio of two quantities in a proportion, the line of the sign $\div$ is omitted, and the two dots (:) are used to imply a division of one quantity by another.

9. EQUALITY is indicated by two short horizontal lines, $=$. Thus, $10 + 6 = 16$ signifies that the sum of 10 and 6 is equal to 16.

NOTE. In writing a proportion, the equality of ratios is usually indicated by four dots (: :).

10. INEQUALITY is indicated by the angle $>$, or $<$, the opening being towards the larger quantity. Thus, $12 + 5 > 14$ signifies that the sum of 12 and 5 is greater

---

How is Multiplication indicated? Division? Equality? Inequality?

than 14; and $14 < 12 + 5$ signifies that 14 is less than the sum of 12 and 5.

11. A PARENTHESIS, ( ), or a VINCULUM, ——, is used to include quantities which are to be considered together, or subjected to the same operation. Thus, $(4+2+6) \times 3$ signifies that the sum of 4, 2, and 6 is to be multiplied by 3; and $\overline{9 - 5} \div 2$ signifies that the difference of 9 and 5 is to be divided by 2.

EXAMPLES.

1. $9 + 16 + 11$ indicates how many? Ans. 36.
2. $17 - 6 + 3$ indicates how many?
3. $25 \times 3 + 6 - 2 =$ how many? Ans. 79.
4. $56 \div 7 + 2$ indicates how many?
5. $\frac{81 + 9}{10} + 20 =$ how many? Ans. 29.
6. $\frac{63 - 13}{25} - 1 =$ how many? Ans. 1.
7. $\frac{16 \times 11}{4} - \frac{32 + 18}{9}$ indicates how many?
8. Find the value of $(17 - 5) \times 8$. Ans. 96.
9. Find the value of $\overline{108 + 12} \div (16 - 4.)$
10. Add $\frac{27 + 6}{11}$ and $\frac{115 + 35}{25} + 41$. Ans. 50.
11. Subtract $\frac{(19 - 7) \times 10}{20}$ from $(81 - 63) \times 6$.
12. Show that $\frac{916 - 788}{64} + 14 > 3 \times 4$.
13. Show that $(1137 - 869) \div 67 < (101 + 37) \div 23$.

How is a Parenthesis or a Vinculum used?

ALGEBRAIC NOTATION.

12. ALGEBRAIC NOTATION is of a mixed character, consisting principally of the figures of Arithmetic and of the letters of the alphabet.

13. FIGURES of Arithmetic are used to represent known quantities and determined values.

14. LETTERS are used to represent any quantity whatever, known or unknown.

15. KNOWN QUANTITIES, or those whose values are given, are generally represented by the first letters of the alphabet, as a, b, c.

UNKNOWN QUANTITIES, or those whose values are to be determined, are generally represented by the last letters of the alphabet, as x, y, z.

16. NUMERICAL QUANTITIES are those represented by figures.

LITERAL QUANTITIES are those represented by letters.

17. FACTORS are quantities which are to be multiplied together.

18. A COEFFICIENT of a quantity is a figure or letter prefixed to it, to show how many times the quantity is to be taken. Thus, in $4a = a + a + a + a$, 4 is the coefficient of a, and indicates that a is taken 4 times; in bx, b is the coefficient of x, and indicates that x is taken b times; and in $5cy$, 5 may be regarded as the coefficient of cy, or $5c$ as the coefficient of y.

When no coefficient of a quantity is written, 1 is understood to be its coefficient. Thus, a is the same as $1a$, and xy is the same as $1xy$.

Of what does Algebraic Notation consist? What are Figures used to represent? Letters? How are Known Quantities represented? Unknown Quantities? Define Numerical Quantities. Literal Quantities. Factors. A Coefficient.

As $4a$ indicates that four a's are to be added together, or that a is to be taken four times, it is evident that a and 4 are to be multiplied together. The expression bx also indicates that x is to be multiplied by b. Hence, when a figure and a letter, or two letters, are separated by no sign, multiplication is understood, and the quantities are to be used as factors. (Art. 7.)

NOTE. This method of expressing multiplication cannot be extended to *figures* separated by no sign. Thus, 82 could not be used to denote the *product* of 8 and 2, because that form is already appropriated in Arithmetic to eighty-two, or the *sum* of 8 tens and 2 units.

In such expressions as $8\ (2+3)$, multiplication is understood, for the figures 8 and 2 are separated by a sign.

19. An EXPONENT is a figure or letter written at the right and above a quantity, to indicate the number of times the quantity is taken as a factor. Thus, $x \times x \times x$, or $x\,x\,x$, may be written x^3, in which 3 is the exponent of x, and indicates that x is taken 3 times as a factor.

If the minus sign is prefixed to an exponent, it indicates that the quantity is to be used as a divisor. Thus $a^2\,b^{-1}\,c^{-3}$ is the same as $\frac{a^2}{b\,c^3}$, and x^{-2} is the same as $\frac{1}{x^2}$.

Unless a parenthesis or vinculum is used, an exponent affects only the single letter or figure to which it is affixed. Thus, in the expression $3ab^2$, the 2 affects only the b. If it were to extend its power to the whole expression, it would be written thus, $(3ab)^2$.

NOTE. It will be observed that the coefficient and exponent both signify how many times a quantity is taken. The one, however, denotes that it is taken as an *additive* quantity, the other as a *factor*. Thus, $5a$ denotes 5 a's added together, or $a+a+a+a+a$, while a^5 denotes 5 a's multiplied together, or $a \times a \times a \times a \times a$.

What does the absence of any sign between two quantities indicate? Define an Exponent. How far does the power of an exponent extend?

20. A POWER of any quantity is the product obtained by taking that quantity one or more times as a factor, and is expressed by an exponent.. Thus,

$a \times a = a^2$, read a square, is the *second* power of a;
$a \times a \times a = a^3$, read a cube, is the *third* power of a;
$a \times a \times a \times a = a^4$, read a fourth, is the *fourth* power of a.

When a quantity has no exponent written, it is understood to be the *first* power. Thus, a is the same as a^1, or the first power of a.

NOTE. The relation of a *power* to a *product* is similar to that of a *product* to a *sum*. The addition of equal quantities, or of a quantity to itself, is multiplication; and the multiplication of equal quantities, or of a quantity by itself, is raising to a power. Thus, $3a$ is either the *sum* of three a's, or the *product* of 3 and a; and a^3 is either the *product* of three a's, or the third *power* of a.

21. A ROOT of any quantity is a factor which, taken a certain number of times, will form that quantity. Thus,

a is the *second* or *square* root of a^2, since $a \times a = a^2$;
a is the *third* or *cube* root of a^3, since $a \times a \times a = a^3$;
a is the *fourth* root of a^4, since $a \times a \times a \times a = a^4$.

22. The RADICAL SIGN, $\surd$, when prefixed to a quantity, indicates that the root is to be taken. Thus,

$\sqrt[2]{a}$ indicates the *second* or *square* root of a;
$\sqrt[3]{a}$ indicates the *third* or *cube* root of a;
$\sqrt[4]{a}$ indicates the *fourth* root of a.

The *index* of the root is the figure or letter written over the radical. Thus, 2 is the index of the square root, 3 of the cube root, and so on.

When the radical has no index over it, 2 is understood. Thus, $\sqrt{a}$ is the same as $\sqrt[2]{a}$.

A *fractional exponent* is also used to indicate a root.

Define a Power. A Root. What does the Radical Sign indicate? Define an Index of the root.

Thus $a^{\frac{1}{2}}$ indicates the square root of a, and $a^{\frac{1}{3}}$ indicates the cube root of a. The numerator of the exponent denotes the power, and the denominator the root. Thus, $b^{\frac{5}{4}}$ indicates the fifth power of the fourth root of b, or the fourth root of the fifth power of b.

ALGEBRAIC EXPRESSIONS.

23. An Algebraic Expression is a quantity written in algebraic language. Thus,

$2a$ is the algebraic expression for 2 times the number a.

$3a^2$ is the algebraic expression for 3 times the square of the number a.

$5a + 7b^3$ is the algebraic expression for 5 times a, augmented by 7 times the cube of b.

24. The Terms of an algebraic expression are its parts connected by the signs $+$ or $-$. Thus,

a and b are the terms of the expression $a + b$;

$2a$, b^2, and $-2ac$, of the expression $2a + b^2 - 2ac$.

25. The Degree of a term is the number of literal factors which it contains. Thus,

$3a$ is of the *first* degree, since it contains but *one* literal factor.

ab is of the *second* degree, since it contains but *two* literal factors.

$5ab^2$ is of the *third* degree, since it contains but *three* literal factors.

The degree of any term is determined by adding the exponents of its several letters. Thus,

$3ab^2c^3$ is of the *sixth* degree, since $1 + 2 + 3 = 6$.

What does a fractional exponent indicate? Define an Algebraic Expression. The Terms of an algebraic expression. Degree of a term.

26. A Monomial is an algebraic expression consisting of only one term; as, $5a$, $7ab$, or $3b^2c$.

27. A Polynomial is an algebraic expression consisting of more than one term; as,

$$a+b, \text{ or } 3a^2+b-5b^3.$$

28. A Binomial is a polynomial of two terms; as,

$$a-b,\ 2a+b^2, \text{ or } 3ac^2-b.$$

29. A Trinomial is a polynomial of three terms; as,

$$a+b+c, \text{ or } ab+c^2-b^3.$$

30. Homogeneous Terms are those of the same degree.

Thus, the terms a^2, $3bc$, $-4x^2$ are homogeneous.

A polynomial is homogeneous when all its terms are homogeneous. Thus, the polynomial $a^3+abc-b^3$ is homogeneous.

31. Positive Terms are those having the *plus* sign; as,

$$+a, \text{ or } +ab^2.$$

When a term has no sign written, it is understood to be positive. Thus, a is the same as $+a$.

32. Negative Terms are those having the *minus* sign; as, $-a$, or $-2bc^2$. This sign should never be omitted.

33. Similar or Like Terms are those containing the same letters, affected by the same exponents.

Thus, $2xy$ and $-7xy$ are similar terms;
also, $3a^2b^3$ and $9a^2b^3$ are similar terms.

34. Dissimilar or Unlike Terms are those containing different letters or exponents.

Thus, ab and ad are dissimilar terms;
also, bx^2y and bxy^2 are dissimilar terms.

35. The Reciprocal of a quantity is 1 divided by that quantity.

Define a Monomial. A Polynomial. A Binomial. A Trinomial. Homogeneous Terms. Positive Terms. Negative Terms. Similar Terms. Dissimilar Terms. A Reciprocal of a quantity.

Thus, the reciprocal of a is $\frac{1}{a}$, and of $x + y$ is $\frac{1}{x+y}$.

Note. The *reciprocal of a fraction* is that fraction inverted. Thus, $\frac{n}{m}$ is the reciprocal of $\frac{m}{n}$.

The following examples will serve for an exercise on the preceding principles.

Examples.

Put in the form of algebraic expressions:—

1. Three times b, added to two times a.
Ans. $2a + 3b$.

2. Three times b, subtracted from five times a.
Ans. $5a - 3b$.

3. The sum of a and b, diminished by c.
Ans. $a + b - c$.

4. The sum of x and two times y, diminished by z.

5. a plus the product of b and c, minus d.
Ans. $a + bc - d$.

6. The sum of a and b multiplied by the difference of c and d. Ans. $(a+b)(c-d)$.

7. Five times b, divided by four times c.
Ans. $\frac{5b}{4c}$.

8. Four times a, divided by three times c.

9. a diminished by b, divided by a multiplied by b.

10. a plus b, multiplied by c into d.
Ans. $(a+b)cd$.

11. Two times a, plus the quotient of b divided by c.

12. Six times a square into b cube, plus three times c square into d cube. Ans. $6a^2b^3 + 3c^2d^3$.

13. a fourth power minus b fifth, divided by a minus b square.

14. Two a square, into a minus b, into c plus d, plus c cube. Ans. $2a^2(a-b)(c+d)+c^3$.

15. Fifteen a cube plus b fifth, divided by a square minus b square, plus two c.

16. The reciprocal of c minus d, plus two a square, minus b cube. Ans. $\frac{1}{c-d}+2a^2-b^3$.

17. The reciprocal of a into b square, minus the reciprocal of a square plus c square.

18. The square root of a, plus the square root of b. Ans. $\sqrt{a}+\sqrt{b}$.

19. The cube root of a, minus b. Ans. $\sqrt[3]{a}-b$.

20. The square root of a minus b. Ans. $\sqrt{a-b}$.

21. The cube root of x, minus the square root of x.

22. Write a polynomial of three terms, with its third term negative.

23. Write a homogeneous binomial of the first degree; a homogeneous trinomial of the third degree, with its second term negative.

INTERPRETATION OF ALGEBRAIC EXPRESSIONS.

36. The INTERPRETATION of an algebraic expression consists in rendering it into arithmetic, by means of the numerical values assigned to its letters.

37. The NUMERICAL VALUE of an algebraic expression is the result obtained by substituting for its letters their numerical values, and then performing the operations indicated.

Thus, the numerical value of

$$4a+3bc-d,$$

What is the Interpretation of an algebraic expression? How is its Numerical Value obtained?

when $a=4$, $b=3$, $c=5$, and $d=2$,
is $$4\times 4+3\times 3\times 5-2=59.$$

Examples.

Interpret and give the numerical values of the following expressions, when

$$a=12,\ b=3,\ c=2,\ d=4,\ m=5,\ n=9.$$

1. $a+b-c+d$. Ans. 17.
2. $ab+c-d$. Ans. 34.
3. $4a-5b+4c-7d$.
4. $(a-b)(c+d)$. Ans. 54.
5. $(6a+b^2)c+d$. Ans. 166.
6. $\frac{a+b}{m}+mn$. Ans. 48.
7. $c^2(a+b)-\frac{a}{d}$.
8. $2c(a-b)-(b+c)d$. Ans. 16.
9. $2a^2c-\frac{a^2}{c}+\frac{a}{c^2}$. Ans. 507.
10. $\frac{a+b^2+c^3}{13}\times\frac{a^2-b^3}{n}$. Ans. 29.
11. $\left(\frac{b^2+c}{11}+a\right)\times(b-c)-d$. Ans. 9.
12. Find the value of c^4-4c^3+3c-6, when $c=4$.
13. If $a=6$, $b=5$, $c=4$, $d=1$, $x=0$, find the value of $7a^2+(b-c)(d-x)$. Ans. 253.
14. If $a=4$, $b=2$, $c=3$, $d=1$, find the value of $15a-7(b+c-d)$. Ans. 32.
15. If $x=3$ and $y=5$, find the value of
$$(9-y)(x+1)+(x+5)(y+7)-112.$$
16. If $b=2$, $c=3$, $d=1$, find the value of
$$\sqrt{8b}+\sqrt{100d}-\sqrt{27c}.$$ Ans. 5.
17. If $a=6$, $b=5$, $c=4$, find the value of
$$2a\sqrt{b^2-ac}+\sqrt{2ac+c^2}.$$ Ans. 20.

18. If $a=2$, $b=3$, $c=4$, find the value of
$\sqrt{27b}-\sqrt[3]{2c}+\sqrt{2a}$. Ans. 9.

19. If $a=10$, $b=8$, $x=12$, $y=4$, find the value of $a+b\sqrt{(x+y)}-(a-b)\sqrt[3]{(x-y)}$. Ans. 38.

AXIOMS.

38. An Axiom is a self-evident truth.

Algebraic operations are based upon definitions and the following axioms: —

1. If the same quantity, or equal quantities, be *added* to equal quantities, the *sums* will be equal.

2. If the same quantity, or equal quantities, be *subtracted* from equal quantities, the remainders will be equal.

3. If equal quantities be *multiplied* by the same quantity, or equal quantities, the *products* will be equal.

4. If equal quantities be *divided* by the same quantity, or equal quantities, the *quotients* will be equal.

5. If the same quantity be both *added* to and *subtracted* from another, the value of the latter will not be changed.

6. If a quantity be both *multiplied* and *divided* by another, the value of the former will not be changed.

7. Quantities which are equal to the same quantity are equal to each other.

8. Like powers and like roots, of equal quantities, are equal.

9. The whole of a quantity is equal to the sum of all its parts.

Define an Axiom? Upon what are algebraic operations based? Repeat the axioms given.

ALGEBRAIC PROCESSES.

39. The PROCESSES of Algebra, in general, are only those of Arithmetic extended, or rendered more comprehensive by the aid of letters taken in combination with figures. (Art. 12.)

The processes of Algebra are employed in the demonstration of theorems and in the solution of problems.

40. A THEOREM is the statement of some relation or property, the truth of which is required to be demonstrated.

41. A PROBLEM is a question proposed for solution, or something to be done.

42. An EQUATION is the expression of equality between two quantities. Thus,

$$x = a - b,$$

is an equation, expressing equality between x and $a - b$.

43. The FIRST MEMBER of an equation is the quantity on the left of the sign of equality; and

The SECOND MEMBER is the quantity on the right of that sign. Thus, in the equation,

$$5x + y = b + c,$$

$5x + y$ is the first member, and $b + c$ is the second.

44. The SOLUTION of a problem or question by Algebra, consists of two parts.

1. *In* STATING THE QUESTION, *by expressing its conditions in the form of an equation.*

2. *In* SOLVING THE EQUATION, *by finding the value of the unknown quantity.*

What is said of the Processes of Algebra? Define a Theorem. A Problem. An Equation. The First Member of an equation. The Second Member. Solution of a problem.

Hence, the solution of the *equation* is the solution of the *problem*.

45. The VERIFICATION of the value found for the unknown quantity, is the process of proving that it will satisfy the conditions of the question.

Thus, in the equation,

$$2x = 9 + 5,$$

if the value of x be found to be 7, it may be verified by substituting 7 for x, and showing that

$$2 \times 7 = 9 + 5.$$

46. To show some of the simpler algebraic forms and processes pertaining to the solution of problems, there are introduced the following

EXAMPLES.

1. The sum of the ages of two boys is 21 years, and the age of the older is twice that of the younger; what is the age of each?

SOLUTION.

Let x = age of the younger;
$2x$ = age of the older;
$3x$ = 21 years.
x = 7 years, the younger;
$2x$ = 14 years, the older.

VERIFICATION. $7 + 14 = 21$.

In this question, if the age of the younger were known, we could, by doubling it, obtain that of the older. The age of the younger, then, may be regarded as the unknown quantity.

We therefore represent the age of the younger by x; then, as the age of the older is twice that of the younger, $2x$ will represent the age of the older; and $x + 2x$, or $3x$, will represent the sum of their ages, which, by the conditions of the question, is 21 years. Hence, if $3x$ equals 21 years, x, the age of the younger, must be one third of 21 years, or 7 years; and $2x$, the age of the older, must be 2 times 7 years, or 14 years.

Define the Verification of the value of an unknown quantity. Explain the operation.

2. John had 45 cents; after spending a part of them, he found he had twice as many left as he had spent; how many cents had he spent? Ans. 15 cents.

3. James and William together have 56 apples, and one has as many as the other; how many has each?

4. A tree 60 feet high was broken at such a point that the part broken off was 3 times the length of the part left standing; required the length of each part.

Ans. Part left standing, 15 ft.; part broken off, 45 ft.

5. The greater of two numbers is 5 times the less, and their sum is 126; required the numbers.

Ans. Less number, 21; greater number, 105.

6. My horse and chaise together are worth $ 340, and the horse is worth 3 times as much as the chaise; what is each worth? Ans. Chaise, $ 85; horse, $ 255.

7. A gentleman divided property, amounting to $ 2500, between his two sons, A and B, and gave B 4 times as much as he gave A; how much did he give to each?

Ans. A, $ 500; B, $ 2000.

8. The sum of three numbers is 72; the second is equal to twice the first, and the third is equal to three times the first; what are the numbers?

SOLUTION.

Let $x =$ first number;
$2x =$ second number;
$3x =$ third number.
$6x = 72.$
$x = 12$, first number;
$2x = 24$, second number;
$3x = 36$, third number.

VERIFICATION. $12 + 24 + 36 = 72.$

We represent the first number by x; then, as the second number is twice the first, $2x$ will represent the second; as the third number is three times the first, $3x$ will represent the third; and $x + 2x + 3x$, or $6x$, will represent the sum of the three numbers, which, by the conditions of the question,

---

Explain the operation.

is 72. Hence, if $6x$ equals 72, x, the first number, must be one sixth of 72, or 12; $2x$, the second number, must be 2 times 12, or 24; and $3x$, the third number, must be three times 12, or 36.

9. It is required to divide $ 300 among A, B, and C, so that B and C may each have twice as much as A. How many dollars will each have?

Ans. A's share, $ 60; B's share, $ 120; C's share, $ 120.

10. Henry bought some apples, pears, and oranges, for 63 cents; he paid for the pears 2 times as much as for the apples, and for the oranges 4 times as much as for the apples; what did he pay for each kind of fruit?

11. The sum of the ages of A, B, and C is 78 years; but B's age is twice that of A, and C's is equal to the sum of A's and B's; what is the age of each?

Ans. A's, 13 years; B's, 26 years; C's, 39 years.

12. A farmer sold a sheep, cow, and horse for $ 180; the cow brought 7 times as much as the sheep, and the horse 4 times as much as the cow; how much did he get for each?

Ans. Sheep, $ 5; cow, $ 35; horse, $ 140.

13. The sum of three numbers is 350; the second is four times the first, and the third is one half the second; what are the numbers?

Ans. First, 50; second, 200; third, 100.

14. John traveled 84 miles in 3 days; he traveled 3 times as far the second day as the first, and half as far the third day as the first two days together; how many miles did he travel each day?

Ans. First, 14 miles; second, 42 miles; third, 28 miles.

15. Three men together contributed for the aid of wounded soldiers, $ 600. A gave a certain sum, B gave 4 times as much, and C gave an amount equal to the difference between what the other two gave. How much did each contribute? Ans. A, $ 75; B, $ 300; C, $ 225.

ADDITION.

47. Addition, in Algebra, is the process of collecting two or more quantities into one equivalent expression, called the sum.

48. In algebraic addition there are three cases, depending upon the similarity and signs of the terms: —

I. When the terms are similar, and have the same sign.

II. When the terms are similar, and have different signs.

III. When the terms are dissimilar, or some similar and others dissimilar.

CASE I.

49. When the terms are similar, and have the same sign.

1. John has 4 books, Edward 6 books, and James 7 books; how many books have they all?

OPERATION.

$$\left.\begin{array}{r} 4 \text{ books,} \\ 6 \text{ books,} \\ 7 \text{ books,} \\ \hline 17 \text{ books,} \end{array}\right\} \text{or,} \left\{\begin{array}{r} 4\,b \\ 6\,b \\ 7\,b \\ \hline 17\,b \end{array}\right.$$

It is evident, by Arithmetic, that the sum of 4 books, 6 books, and 7 books is 17 books.

Now, instead of writing the word *books*, we may simply use the letter b; or we may represent one book by the letter b; then, $4\,b$ will represent 4 books, $6\,b$ will represent 6 books, and $7\,b$ will represent 7 books; and since 4 books + 6 books + 7 books = 17 books, $4\,b + 6\,b + 7\,b = 17\,b$.

2. Let it be required to find the sum of $-4\,b$, $-6\,b$, and $-7\,b$.

Define Addition in Algebra. How many Cases in algebraic addition? Name them. Explain the first operation under Case I.

OPERATION.

$$\begin{array}{r} -\ 4b \\ -\ 6b \\ -\ 7b \\ \hline -17b \end{array}$$

In the same manner as in the preceding operation, $(-4b)+(-6b)+(-7b)=-17b$; since, whatever b may represent, it is taken, in the first term, -4 times; in the second, -6 times; and in the third, -7 times; or, in all, -17 times.

Hence, when the terms are similar and have the same sign:

RULE.

Add the coefficients, and to their sum, with the common sign, annex the common letter or letters.

Note. It must be remembered, that when a quantity has no coefficient written, 1 is understood (Art. 18), and that when a term has no sign written, + is understood (Art. 31).

Examples.

(3.)	(4.)	(5.)	(6.)
$2a$	$4ax$	$2xy$	$-3abc$
$3a$	$2ax$	xy	$-\ abc$
$5a$	ax	xy	$-5abc$
a	$6ax$	$7xy$	$-2abc$
$7a$	$5ax$	$2xy$	$-8abc$
$6a$	$2ax$	xy	$-4abc$
$24a$	$20ax$	$14xy$	$-23abc$

(7.)	(8.)	(9.)	(10.)
$-4bx$	$6mn^2$	$2a+b$	$3c^2d-a^3c$
$-7bx$	$5mn^2$	$a+b$	c^2d-a^3c
$-3bx$	mn^2	$4a+b$	$2c^2d-a^3c$
$-2bx$	$3mn^2$	$7a+b$	$5c^2d-a^3c$
$-5bx$	$2mn^2$	$3a+b$	c^2d-a^3c

Explain the operation. Repeat the Rule. The Note.

11. What is the sum of $-6n$, $-4n$, $-n$, $-8n$, and $-12n$? Ans. $-31n$.

12. What is the sum of $5x$, $2x$, x, $3x$, $4x$, $6x$, x, and $8x$? Ans. $30x$.

13. What is the sum of $2x+3y$, $x+8y$, $3x+y$, $6x+2y$, $x+4y$, and $4x+y$? Ans. $17x+19y$.

14. What is the sum of $7a^2-b$, $3a^2-3b$, $6a^2-2b$, $2a^2-b$, $4a^2-6b$, and a^2-4b? Ans. $23a^2-17b$.

CASE II.

50. When the terms are similar, and have different signs.

1. Let it be required to add $+8a$, $-5a$, $+7a$, and $-3a$.

OPERATION.

$$\begin{array}{r} +8a, \\ -5a, \\ +7a, \\ \underline{-3a,} \\ +7a. \end{array}$$

Since the terms to be added are some positive and others negative, in finding their sum regard must be paid to their signs. Now, the signs, + and − indicate, not only opposite processes, but may be regarded as used to denote opposite qualities, effects, or conditions of quantities.

Thus, if a merchant's *gains* are indicated by +, his *losses* will be indicated by −; if distance *north* be reckoned +, distance *south* will be −, and so on. Hence, two equal quantities, of which one is positive and the other negative, will exactly *balance*, or *cancel* each other.

Now, in the example, $+8a+7a=+15a$; and $-5a-3a$, or $(-5a)+(-3a)=-8a$. But $-8a$ cancels $+8a$ in the quantity $+15a$, which leaves $+7a$ for the sum of the quantities.

2. A merchant having a certain capital, in the first quarter of the year gained $6a$ dollars, and in the second quarter gained $5a$ dollars, but in the third and fourth

---

Explain the first operation.

quarters lost $7a$ and $9a$ dollars. What was the result of the business at the end of the year?

OPERATION.

$$\begin{array}{r} +6a, \\ +5a, \\ -7a, \\ \underline{-9a,} \\ -5a. \end{array}$$

We indicate the gains as positive, and their opposite, the losses, as negative.

The sum of $-9a$ and $-7a$ is $-16a$, and the sum of $+5a$ and $+6a$ is $+11a$. But $+11a$ cancels $-11a$ in the quantity $-16a$, which leaves $-5a$, or a loss of $5a$ dollars.

From the preceding operations, it appears that,

The ALGEBRAIC SUM *of a positive and a negative quantity is numerically the* DIFFERENCE *of the two quantities, with the sign of the greater prefixed.*

Hence, when the terms are similar, and have different signs:

RULE.

Add the coefficients of the positive terms, and also the coefficients of the negative terms, and to the difference of these sums, with the sign of the greater, annex the common letter or letters.

EXAMPLES.

(3.)	(4.)	(5.)	(6.)
$3a$	$4ax$	$2bx + 3by$	$3x^2 - 4xy^3$
$5a$	$-2ax$	$3bx - 2by$	$x^2 + xy^3$
$-2a$	$3ax$	$-5bx + 4by$	$-4x^2 - 3xy^3$
$7a$	$7ax$	$4bx - by$	$2x^2 + 2xy^3$
$-4a$	$12ax$	$-6bx + 7by$	$-x^2 - xy^3$
$9a$		$-2bx + 11by$	

7. What is the sum of $a - b$, $-2a - 7b$, $7a - 2b$, $-3a + 3b$, $-8a + b$, and $a + 7b$? Ans. $-4a + b$.

Explain the second operation. What is the Algebraic Sum of a positive and a negative quantity? Repeat the Rule.

8. What is the sum of $5\,cd^2 + 7\,a^2b$, $-3\,cd^2 - 5\,a^2b$, $9\,cd^2 - 10\,a^2b$, $-4\,cd^2 + a^2b$? Ans. $7\,cd^2 - 7\,a^2b$.

CASE III.

51. When the terms are dissimilar, or some similar, and others dissimilar.

1. What is the sum of $2\,a$, $5\,b$, and $-ac$?

OPERATION.

$$2\,a + 5\,b - ac.$$

If the given terms were similar, the addition could be performed by uniting them into one (Art. 50); but the terms being dissimilar, we can only add them by writing them one after the other, with their respective signs; which gives $2\,a + 5\,b - ac$.

2. What is the sum of $3\,a + b$, $-2\,a - 2\,b$, and $6\,a + 3\,b - 2\,c$?

OPERATION.

$$\begin{array}{rrr} 3\,a & + b & \\ -2\,a & - 2\,b & \\ 6\,a & + 3\,b & - 2\,c \\ \hline 7\,a & + 2\,b & - 2\,c \end{array}$$

We write similar terms in the same column, for convenience in performing the operation.

Beginning at the left, we find $+3\,a - 2\,a + 6\,a = 7\,a$, which we write under the column added; and $+b - 2b + 3\,b = +2\,b$, which we write under the column added; and there being no term similar to $-2\,c$, we write it, with its proper sign, after the other terms obtained, and have as the entire sum, $7\,a + 2\,b - 2\,c$.

Hence, since this case clearly includes the two preceding cases, for the addition of algebraic quantities, the following

GENERAL RULE.

Write similar terms, with their proper signs, in the same column.

Add each column, and to the results obtained annex the dissimilar terms, with their proper signs.

Explain the first operation. The second. Repeat the Rule.

Note. It is immaterial in what order terms connected by + and − may stand, provided each term has its proper sign. Thus, $-b+a$ is the same as $a-b$.

It is, however, more common to commence a polynominal with a positive term, unless there is a special reason for some other arrangement.

EXAMPLES.

(3.)

$$\begin{array}{r}
3ax - by + x^2 \\
ax + 2by \\
4ax - 3by + 2x^2 \\
5by - x^2 \\
\hline
8ax + 3by + 2x^2
\end{array}$$

(4.)

$$\begin{array}{r}
2x + 8y^3 \\
6x - 4y^3 - 2x^3 \\
3y^3 + x^3 \\
-4x + y^3 - 3x^3 \\
\hline
4x + 8y^3 - 4x^3
\end{array}$$

5. What is the sum of $-ab^2 - cd^2$, $-ab^2 + cd^2$, $-3ab + cd^2$, and $5ab^2 + cd^2$?

6. What is the sum of $3x - 7y + 2z$, $4y + 6z - x$, $-3z - 2y + a$, and $4x + 3z - y$?

Ans. $6x - 6y + 8z + a$.

7. What is the simplest equivalent expression for $-5ax + 2by - 7$, $3by + 18 - 4z$, $4ax - 9 - by$, and $26 + 3ax - 2by$? Ans. $2ax + 2by + 28 - 4z$.

8. Add $x^3 + ax^2 + bx + 2$, $3x^3 - 4ax^2 - 6bxy + 7$, and $3x^3 - 3ax^2 - 7bx - 19$.

Ans. $7x^3 - 6ax^2 - 6bx - 6bxy - 10$.

9. Find the sum of $8a^2x^2 - 3ax$, $7ax - 5xy$, $-5ax + 9xy - b^2c^3$, and $2a^2x^2 + xy$.

Ans. $10a^2x^2 - ax + 5xy - b^2c^3$.

52. Similar quantities, of any kind, may be added by taking the algebraic sum of their coefficients; and quantities inclosed in a parenthesis may be considered as one quantity. (Art. 11.)

1. What is the sum of $3(a+b)$, $5(a+b)$, and $8(a+b)$?

Repeat the Note. How may quantities in a parenthesis be considered?

OPERATION.

$$\begin{array}{r} 3\,(a+b) \\ 5\,(a+b) \\ 8\,(a+b) \\ \hline 16\,(a+b) \end{array}$$

We consider $(a+b)$ as a single quantity. Then, since 3 times, 5 times, and 8 times any quantity, will equal 16 times that quantity, $3\,(a+b)+5\,(a+b)+8\,(a+b)=16\,(a+b)$.

2. Required the sum of $5\,(a+x)$, $6\,(a+x)$, $8\,(a+x)$, $3\,(a+x)$, and $(a+x)$. Ans. $23\,(a+x)$.

3. Required the sum of $3\,(x^2-a)$, $2\,(x^2-a)$, $-(x^2-a)$, $6\,(x^2-a)$, and (x^2-a).

4. Find the sum of $4\sqrt{a-x}$, $3\sqrt{a-x}$, $-7\sqrt{a-x}$, $2\sqrt{a-x}$, and $\sqrt{a-x}$. Ans. $3\sqrt{a-x}$.

5. Find the sum of $7\,y-4\,(a+b)$, $6\,y+2\,(a+b)$, $2\,y+(a+b)$, and $y-3\,(a+b)$.

Ans. $16\,y-4\,(a+b)$.

6. Find the sum of $2\,(x-y)^2$, $3\,(x-y)^2$, $(x-y)^2$, $-(x-y)^2+(x+y)$, and $(x+y)$.

Ans. $5\,(x-y)^2+2\,(x+y)$.

53. When dissimilar terms have a *common factor* (Art. 17, 18), they may be added by annexing that factor to the sum of its coefficients, inclosed in a parenthesis.

The quantity whose coefficients are added will then be considered as a single quantity.

1. Required the sum of $a\,x^2$, $b\,x^2$, and $c\,x^2$.

OPERATION.

$$\begin{array}{r} a\,x^2 \\ b\,x^2 \\ c\,x^2 \\ \hline (a+b+c)\,x^2 \end{array}$$

The terms, although dissimilar, have a common factor, x^2, which as such we use in the addition. Then, since a times, b times, and c times x^2 will equal x^2 multiplied by the sum of a, b, and c, we indicate the addition of a, b, and c, which are dissimilar, and,

---

Explain the operation. How may dissimilar terms having a common factor be added? Explain the operation.

inclosing the sum in a parenthesis, write it as the coefficient of x^2, and thus obtain the sum required.

2. What is the sum of bx, abx, and $2cx$?
Ans. $(b+ab+2c)x$.

3. Find the sum of $3ay$, $-cy$, and $-2ay$.
Ans. $(a-c)y$.

4. What is the sum of $(a+b)x$ and $(a-c)x$?
Ans. $(2a+b-c)x$.

5. What is the sum of $(a+b)x$, $2cx$, and $2x$?
Ans. $(a+b+2c+2)x$.

6. What is the sum of $ax+b$ and $cx+d$?
Ans. $(a+c)x+b+d$.

7. Add $ax+7m$, $7ax-3m$, and $bx+4m$.
Ans. $(8a+b)x+8m$.

8. Find the sum of ax^2+bx and cx^2-dx.
Ans. $(a+c)x^2+(b-d)x$.

SUBTRACTION.

54. Subtraction, in Algebra, is the process of finding the difference between two algebraic quantities.

The *Subtrahend* is the quantity subtracted.

The *Minuend* is the quantity from which it is subtracted.

The *Difference,* or *Remainder,* is the quantity left after the subtraction is performed.

1. If I have $8a$ dollars and give away $3a$ dollars, how many shall I have left?

Define Subtraction in Algebra. Subtrahend. Minuend. Difference.

OPERATION.

$$\begin{array}{r} 8a \\ \underline{3a} \\ 5a \end{array}$$

Since 8 times any quantity less 3 times the same quantity will equal 5 times the quantity, $8a - 3a = 5a$.

It will be noticed that in expressing the difference between the quantities, the sign of the subtrahend is changed from $+$ to $-$.

2. Let it be required to take $8b$ from $5b$.

OPERATION.

$$\begin{array}{r} 5b \\ \underline{8b} \\ -3b \end{array}$$

We cannot, *numerically*, take 8 times any quantity from 5 times the same quantity. If we take $5b$ from $5b$, nothing will remain; there is yet, however, a quantity, $3b$, *to be subtracted*, with nothing to take it from, which we indicate by $-3b$.

As in the previous example, the expression for the difference is $5b - 8b$, and this expression reduced to its simplest form, according to the rules of Addition (Art. 50), is $-3b$.

3. A thermometer was observed to stand at one time at 17 degrees above zero, and at another time at 5 degrees below; required the difference in range.

OPERATION.

$$\begin{array}{r} +17d \\ \underline{-\ \ 5d} \\ +22d \end{array}$$

For degree we write d, and indicate the range above zero as positive, and that below as negative. Then the difference of range, which evidently must equal the degrees above zero plus those below, will be $17d + 5d = 22d$, or 22 degrees.

Here, as in the former operations, in expressing the difference, the sign of the subtrahend is changed, but in this case from $-$ to $+$.

It will also be observed that the *algebraic difference* between two quantities may be numerically greater than either of them.

4. Let it be required to take $b + c$ from a.

Explain the first operation. The second. The third.

OPERATION.

$$\begin{array}{l} a \\ \underline{b + c} \\ a - b - c \end{array}$$

If we take b from a, the remainder is obviously $a - b$. But a is to be diminished by c, as well as b, consequently the true remainder will be $a - b$ diminished by c, or $a - b - c$.

5. Let it be required to take $b - c$ from a.

OPERATION.

$$\begin{array}{l} a \\ \underline{b - c} \\ a - b + c \end{array}$$

If we take b from a, we obtain $a - b$. But, in doing this, we subtract c too much, consequently the true remainder will be $a - b$ increased by c, or $a - b + c$.

Now, in performing each of the above operations, we have simply changed the signs of the subtrahend, and then added it to the minuend.

In like manner may any quantity whatever be subtracted from another.

Hence, for the subtraction of algebraic quantities, the

GENERAL RULE.

Conceive the signs of all the terms of the subtrahend to be changed, from + to —, or from — to +, and then proceed as in addition.

Note. Subtraction may be proved, as in Arithmetic, by adding the remainder, or difference, to the subtrahend. If the work is right, the sum should equal the minuend.

Examples.

(6.)	(7.)	(8.)	(9.)	(10.)
$5a$	$-12x$	$21y$	$-3xy$	$14y$
$2a$	$-4x$	$-21y$	$+7xy$	$3y$
$3a$	$-8x$	$42y$	$-10xy$	

Explain the fourth operation. The fifth operation. Repeat the general Rule. How may subtraction be proved?

(11.)

$$\begin{array}{r} -\ 7a \\ -16a \\ \hline \end{array}$$

(12.)

$$\begin{array}{r} 4ax^2 \\ ax^2 \\ \hline \end{array}$$

(13.)

$$\begin{array}{r} -8x^2y^3 \\ +8x^2y^3 \\ \hline \end{array}$$

(14.)

$$\begin{array}{r} 3x-2a \\ 4x+\ a \\ \hline -x-3a \end{array}$$

(15.)

$$\begin{array}{r} 27a-\ 7b+\ 5x \\ 13b-\ 7x \\ \hline 27a-20b+12x \end{array}$$

(16.)

$$\begin{array}{r} -\ bx^2+cx-\ 5d \\ bx^2+cx-12d-19 \\ \hline -2bx^2\qquad+\ 7d+19 \end{array}$$

17. From $12b$ take $14b$. Ans. $-2b$.
18. From $27a$ take $-9a$. Ans. $36a$.
19. From $-8c$ take $-8c$.
20. From $5x$ take $-7x$. Ans. $12x$.
21. From $-11d$ take $4d$.
22. From $a+b$ take $-a$. Ans. $2a+b$.
23. From $a+b$ take $a-b$. Ans. $2b$.
24. From $a-b$ take $a+b$. Ans. $-2b$.
25. From $a-b$ take $b-a$. Ans. $2a-2b$.
26. From $5xy$ take $3xy-3$. Ans. $2xy+3$.
27. From $a+b+c$ take $a-b-c$. Ans. $2b+2c$.
28. From x take $x+y$. Ans. $-y$.
29. From $x+5$ take $y-2$. Ans. $x-y+7$.
30. From a^2b take ab^2. Ans. a^2b-ab^2.
31. From $16a^2b^2$ take $-15a^2b^2$. Ans. $31a^2b^2$.
32. From $2x^2-y^2$ take $-2x^2-y^2$. Ans. $4x^2$.
33. From $6(a+b)$ take $3(a+b)$. Ans. $3(a+b)$.
34. From $4(a-b)$ take $-5(a-b)$.
Ans. $9(a-b)$.
35. Subtract $3xy-x^3-7a$ from $5xy+2x^3+2a$.
Ans. $2xy+3x^3+9a$.

36. Subtract $3abx - 7$ from $6abx + 12 - 3xy$.
Ans. $3abx + 19 - 3xy$.

37. Subtract $bx^2 + cx - 12d$ from $ax^3 - bx^2 + cx$.
Ans. $ax^3 - 2bx^2 + 12d$.

38. From $3a + b + c - d$, take $3a + b - 18$.
Ans. $c - d + 18$.

39. From $5x - b$ take $-2xy + b$.
Ans. $5x + 2xy - 2b$.

40. From $3a(a - y) + 4by + a^3$, take $2a(a - y) - 7by + 4a^3$. Ans. $a(a - y) + 11by - 3a^3$.

41. If the minuend is $a^3 + 3b^2c + ab^2 - abc$, and the subtrahend is $ab^2 - abc + b^2$, what will be the difference? Ans. $a^3 + 3b^2c - b^2$.

42. Subtract $8a + 4b - 5c - 2x$ from $-6a - 4b - 12c + 12x$. Ans. $-14a - 8b - 7c + 14x$.

43. From $2ab + b^2 - 4c + bc$, take $3ab + 2b^2 - c - 3bc + 4b^2$. Ans. $-ab - 3c + 4bc - 5b^2$.

55. The subtraction of a polynomial from any quantity may be *indicated,* without performing the operation, by writing after the minuend the subtrahend, inclosed in a parenthesis, with the negative sign prefixed.

Thus, the subtraction of $a^2 + b^2 - c$ from $5a^2$ is indicated by the expression,

$$5a^2 - (a^2 + b^2 - c).$$

Remove the parenthesis, and, since the sign $-$ prefixed indicated that all the included terms were to be subtracted from $5a^2$, we change their signs, and have

$$5a^2 - a^2 - b^2 + c, \text{ or } 4a^2 - b^2 + c.$$

Conversely, the expression $5a^2 - a^2 - b^2 + c$ may be transformed to its previous equivalent form of expression, $5a^2 - (a^2 + b^2 - c)$, by changing the signs of the last

---

How may the subtraction of a polynominal be indicated?

three terms and inclosing them in a parenthesis, with the sign — prefixed. Hence,

The signs of all the terms of a quantity must be changed when the quantity is inclosed in a parenthesis, with the sign — prefixed; also, when a quantity is taken out from such a parenthesis, its signs must be changed.

NOTE. It must not be forgotten that, in such expressions as $5a^2 - (a^2 + b^2 - c)$, the sign of a^2 is really plus, as no sign is expressed. The sign — before the parenthesis belongs to $a^2 + b^2 - c$, *as a whole.*

1. Indicate the subtraction of $a + b$ from x.
Ans. $x - (a + b)$.

2. What is the value of $x - (a + b)$?
Ans. $x - a - b$.

3. Place the last two terms of $x - a - b$ in a parenthesis, without changing the value expressed.
Ans. $x - (a + b)$, or $x + (- a - b)$.

4. What is the value of $a - (b - c - d + e)$?
Ans. $a - b + c + d - e$.

5. Indicate the subtraction of $5\,a^2 + b^2$ from $-6\,a^2 - b$.
Ans. $-6\,a^2 - b - (5\,a^2 + b^2)$.

6. Reduce the expression $-6\,a^2 - b - (4\,a^2 + b^2)$ to its simplest form.
Ans. $-10\,a^2 - b - b^2$.

7. What is the value of $a^3 - b^3 - (3\,a^2\,b - 3\,a\,b^2)$?
Ans. $a^3 - b^3 - 3\,a^2\,b + 3\,a\,b^2$.

8. Place the last three terms of $a^2\,b + x\,y^3 - a\,c - 7\,a\,b - 6 + 9\,x^3\,y^2$ in a parenthesis, with the negative sign prefixed, without changing the value expressed.
Ans. $a^2\,b + x\,y^3 - a\,c - (7\,a\,b + 6 - 9\,x^3\,y^2)$.

9. What is the value of $10\,a^2 - (-4\,a^2 + b^2 - c^2)$?

How are the signs of a quantity affected by inclosing it in a parenthesis with the negative sign prefixed? By taking it out?

56. When dissimilar terms have a *common factor*, their difference may often be conveniently expressed by annexing that factor to the difference of its coefficients, inclosed in a parenthesis.

The coefficients whose subtraction is expressed will then be considered a single quantity.

1. Required to take dx from cx.

OPERATION.

$$\begin{array}{r} cx \\ dx \\ \hline (c-d)\,x \end{array}$$

The two terms have a common quantity, x, which as such we use in the subtraction.

Then, since d times x taken from c times x will leave c times x less d times x, we have the expression $(c-d)\,x$.

2. From $3xy^2$ take x^3y. Ans. $(3xy - x^3)\,y$.

3. From $7abc^2$ take adc^2. Ans. $(7ab - ad)\,c^2$.

4. From $ay + by$ take $cy + by$. Ans. $(a - c)\,y$.

5. From $ax - b$ take $cx - d$. Ans. $(a - c)\,x - b + d$.

6. From $ay - by + cy$ take $y + ay - by$. Ans. $(c - 1)\,y$.

7. Subtract $cx^2 - cx$ from $ax^2 + bx$. Ans. $(a - c)\,x^2 + (b + c)\,x$.

8. From $ax^2 + mxy + nx + b$, take $cx^2 - dxy + ex - z$. Ans. $(a - c)\,x^2 + (m + d)\,xy + (n - e)\,x + b + z$.

MULTIPLICATION.

57. Multiplication is the process of taking one quantity as many times as there are units in another quantity.

The *Multiplicand* is the quantity to be multiplied or taken.

How may the difference of dissimilar terms often be expressed? Explain the operation. Define Multiplication. The Multiplicand.

The *Multiplier* is the quantity by which we multiply.

The *Product* is the result of the operation.

The multiplicand and multiplier are often called *factors*.

58. *The product of factors is the same, in whatever order they are taken.*

For, the product contains one factor as many times as there are units in the other. Thus, the product of $a \times b$, or $b \times a$, will be $a b$ units, since b taken a times, is the same as a taken b times. Let $a = 4$ and $b = 3$, we have 4×3, or 3×4, equal to 12.

NOTE. Numerical factors are usually placed before literal ones, as coefficients, and letters are most frequently placed in the order of the alphabet.

59. *The product of two factors having* LIKE *signs is* POSITIVE; *and the product of factors having* DIFFERENT *signs is* NEGATIVE. Thus,

1. Let it be required to find the product of $+a$ by $+b$.

Now, a is to be taken as many times as there are units in b, and as the *sum* of any number of positive quantities is positive, the product, $a b$, must be positive, or $+a b$. (Art. 20, Note.) If $b = 4$, the product of a by b may be represented thus, $a \times 4 = a + a + a + a = 4 a$.

2. Let it be required to find the product of $-a$ by $+b$.

Here we must take $-a$ as many times as there are units in b, and as the *sum* of any number of negative quantities is negative, the product must be negative, or $-a b$. If $b = 4$, the product of $-a$ by b may be represented thus, $(-a) \times 4 = (-a) + (-a) + (-a) + (-a) = -a - a - a - a = -4 a$.

3. Let it be required to find the product of $+a$ by $-b$.

Define the Multiplier. The Product. What are called factors? Does the order in which factors are taken affect the product? What is the product when factors have like signs? What is the product when factors have different signs?

The negative multiplier, $-b$, indicates that a is to be taken as many times as there are units in b, but it is to be *subtracted*, rather than added. Hence, as a positive quantity becomes negative by subtraction, the product must be negative, or $-ab$. If $b = 4$, the product of a by $-b$ may be represented thus, $a \times (-4) = -a - a - a - a = -4a$.

4. Let it be required to find the product of $-a$ by $-b$.

Here we must take $-a$ as many times as there are units in b, and *subtract*; and as a negative quantity becomes positive by subtraction, the product must be positive, or ab. If $b = 4$, the product of $-a$ by $-b$ may be represented thus, $(-a) \times (-4) = -(-a) - (-a) - (-a) - (-a) = a + a + a + a = 4a$.

NOTE. If any difficulty is experienced in conceiving quantities to be independently *additive* or *subtractive*, they may be regarded as added to, or subtracted from, 0, the neutral point, or starting-point, of all positive and negative quantities.

60. From the foregoing discussion it will be noticed, in brief, that in multiplication of algebraic quantities,

Like signs produce $+$, *and unlike signs produce* $-$.

61. In multiplication of algebraic quantities, there will be three cases: —

I. When both factors are monomials.
II. When one factor is a polynomial.
III. When both factors are polynomials.

CASE I.

62. When both factors are monomials.

1. If a man earn $7a$ dollars in 1 week, how much will he earn in $2b$ weeks?

In Multiplication of algebraic quantities, what do like signs produce? Unlike signs? How many cases of algebraic multiplication?

OPERATION.

$$\begin{array}{r} 7\,a \\ \underline{2\,b} \\ 14\,a\,b \end{array}$$

Since the factors in multiplication may be taken in any order (Art. 58), $7\,a \times 2\,b$ is the same as $7 \times 2 \times a \times b$; then, 2 times $7 = 14$; b times $a = ab$; and 14 times $a\,b = 14\,a\,b$, the required result.

2. Required the product of $2\,a^3$ by a^2.

OPERATION.

$$\begin{array}{r} 2\,a^3 \\ \underline{a^2} \\ 2\,a^5 \end{array}$$

Since the exponent of a quantity indicates the number of times the quantity is taken as a factor (Art. 19), $2\,a^3$ is the same as $2\,a\,a\,a$; and a^2 is the same as $a\,a$; then, $a\,a$ times $2\,a\,a\,a = 2\,a\,a\,a\,a\,a$, or $2\,a^5$. The exponent, 5, in the result might have been obtained at once, by taking the sum of the exponents, 3 and 2, of the common letter a. Hence,

The exponent of a letter in the product is equal to the sum of its exponents in the factors.

From the preceding examples and illustrations of multiplication of monomials is derived the following

RULE.

Multiply the numerical coefficients of the two factors together, and annex to the result the letters of both quantities, giving to each letter an exponent equal to the sum of its exponents in the two factors.

Make the product positive, when the factors have like signs, and negative when they have different signs.

EXAMPLES.

(3.)	(4.)	(5.)	(6.)
$3\,a$	$-\,4\,x$	$-\,6\,a$	$12\,x$
$2\,b$	$-\,3\,y$	$+\,2\,b$	$-\,3\,a$
$6\,a\,b$	$12\,x\,y$	$-\,12\,a\,b$	$-\,36$ [illegible]

Explain the first operation under Case I. The second operation. To what is the exponent of a letter in the product equal? Repeat the Rule.

(7.)	(8.)	(9.)	(10.)
$4x$	$+17a$	$-3a$	$-7b$
x	$-\ 2b$	$-4b$	$+4c$

(11.)	(12.)	(13.)	(14.)
$3m^2$	$7x^5$	$-5a^2$	$8x^3y$
$-2m^3$	x^2	$+2a^3$	$5xy$
$-6m^5$	$7x^7$	$-10a^5$	$40x^4y^2$

15. Multiply $8ab$ by $-3c$. Ans. $-24abc$.
16. Multiply $-x^2y$ by $2ax$. Ans. $-2ax^3y$.
17. Multiply $20mn^3$ by $3mn$. Ans. $60m^2n^4$.
18. Multiply $7a^5c$ by $2ab$. Ans. $14a^6bc$.
19. Multiply $-5a^2x^3$ by $-3a^3x$. Ans. $15a^5x^4$.
20. Multiply $-3bcd$ by $-bcd$. Ans. $3b^2c^2d^2$.

Note. Any number of terms inclosed in a parenthesis may be regarded as a monomial.

21. Multiply $a(x+y)$ by b. Ans. $ab(x+y)$.
22. Multiply $2(a+b)$ by a^2. Ans. $2a^2(a+b)$.
23. Multiply $(x-y)^2$ by a. Ans. $a(x-y)^3$.
24. Multiply $(a+b)^2$ by $(a+b)$. Ans. $(a+b)^3$.
25. Multiply $-a(a+y)^3$ by $2a(a+y)^2$.
Ans. $-2a^2(a+y)^5$.
26. Find the product of $2x^m$ by x. Ans. $2x^{m+1}$.
27. Multiply y^m by y^n. Ans. y^{m+n}.
28. Multiply $(a+b)^m$ by $(a+b)^3$.
Ans. $(a+b)^{m+3}$.
29. Find the product of $a^2(x-y)^m$ by $a^m(x-y)^3$.
Ans. $a^{2+m}(x-y)^{m+3}$.

CASE II.

63. When one factor is a polynomial.

1. Required the product of $a+b-c$ multiplied by c.

OPERATION.

$$\begin{array}{l} a+b-c \\ c \\ \hline ac+bc-c^2 \end{array}$$

Since the whole expression $a+b$ is to be multiplied by c, it is evident that each term is to be taken c times; c times $a = ac$; c times $b = bc$; c times $-c = -c^2$; and these partial products, connected with their proper signs, give $ac+bc-c^2$, the required product. Hence the

RULE.

Multiply each term of the multiplicand separately by the multiplier, and connect the partial products by their proper signs.

EXAMPLES.

(2.)

$$\begin{array}{l} 7y+b \\ 4a \\ \hline 28ay+4ab \end{array}$$

(3.)

$$\begin{array}{l} 4a+4x \\ -3n \\ \hline -12an-12nx \end{array}$$

(4.)

$$\begin{array}{l} 6x^2-a \\ 4b \\ \hline 24bx^2-4ab. \end{array}$$

(5.)

$$\begin{array}{l} 7x+4y+a^2 \\ x^2 \\ \hline \end{array}$$

(6.)

$$\begin{array}{l} 5ab-a^2x+x \\ -4ax^2 \\ \hline \end{array}$$

7. Multiply $4a^2b^2+3xy-ac$ by $-a^2$.
Ans. $-4a^4b^2-3a^2xy+a^3c$.

8. Multiply $4a^2y-6y+5x^2$ by 2.
Ans. $8a^2y-12y+10x^2$.

9. Multiply a^2-ax+x^2 by ab.
Ans. $a^3b-a^2bx+abx^2$.

Explain the operation under Case II. Repeat the Rule.

10. Multiply $-m-n-a-c$ by $-m$.
Ans. $m^2+mn+am+cm$.

11. Find the product of $3a+b^2+a^2x-b$ by $4a^2$.
Ans. $12a^3+4a^2b^2+4a^4x-4a^2b$.

12. Find the product of $2x^2y-3xy^2-y^3$ by $5axy$.
Ans. $10ax^3y^2-15ax^2y^3-5axy^4$.

CASE III.

64. When both factors are polynomials.

1. Required the product of $3a+2b$ by $a+b$.

OPERATION.

$$\begin{array}{llll}
3a & +2b & & \\
a & +b & & \\
\hline
3a^2 & +2ab & & \\
 & 3ab & +2b^2 & \\
\hline
3a^2 & +5ab & +2b^2 &
\end{array}$$

Since the multiplicand must be taken as many times as there are units in $a+b$ (Art. 57), it is evident that $3a+2b$ must be taken a times plus b times; a times $3a+2b$ $=3a^2+2ab$; b times $3a+2b$ $=3ab+2b^2$; and the sum of these partial products is $3a^2+5ab+2b^2$; the required product. Hence the following

RULE.

Multiply each term of the multiplicand by each term of the multiplier separately, and add the partial products.

EXAMPLES.

(2.)

$$\begin{array}{rrr}
4a+ & 3b & \\
3a+ & b & \\
\hline
12a^2+ & 9ab & \\
 & 4ab & +3b^2 \\
\hline
12a^2+ & 13ab & +3b^2
\end{array}$$

(3.)

$$\begin{array}{rrr}
5x+ & 3y & \\
x- & 2y & \\
\hline
5x^2+ & 3xy & \\
 & -10xy & -6y^2 \\
\hline
5x^2- & 7xy & -6y^2
\end{array}$$

Explain the operation under Case III. Repeat the Rule.

(4.)

$$\begin{array}{l} 3a^2 - 2y \\ \underline{\ \ x + \ y} \end{array}$$

(5.)

$$\begin{array}{l} a - b \\ \underline{a - b} \end{array}$$

(6.)

$$\begin{array}{l} 5ax + 3x \\ \underline{3ax + 2x} \end{array}$$

(7.)

$$\begin{array}{llll} a^4 + a^2c^2 + c^4 & & & \\ \underline{a^2 - c^2} & & & \\ a^6 + a^4c^2 + a^2c^4 & & & \\ \quad - a^4c^2 - a^2c^4 - c^6 & & & \\ \hline a^6 \qquad\qquad\qquad\qquad - c^6 & & & \end{array}$$

(8.)

$$\begin{array}{llllll} a^m & - \quad a^n & + a^2 & & & \\ a^m & - \quad a^n & & & & \\ \hline a^{2m} & - \quad a^{m+n} & + a^{2+m} & & & \\ & - \quad a^{m+n} & & + a^{2n} & - a^{2+n} & \\ \hline a^{2m} & - 2a^{m+n} & + a^{2+m} & + a^{2n} & - a^{2+n} & \end{array}$$

9. Multiply $3x + 2y$ by $2x - 3y$.
Ans. $6x^2 - 5xy - 6y^2$.

10. Multiply $5a^2 + 3x$ by $5a^2 + 3x$.
Ans. $25a^4 + 30a^2x + 9x^2$.

11. Multiply $a + 2x$ by $a - 3x$.
Ans. $a^2 - ax - 6x^2$.

12. Multiply $3a - x$ by $2a + 4x$.
Ans. $6a^2 + 10ax - 4x^2$.

13. Multiply $x + y$ by $x + y$. Ans. $x^2 + 2xy + y^2$.

14. Multiply $x - y$ by $x + y$. Ans. $x^2 - y^2$.

15. Multiply $a^2 + ab + b^2$ by $a - b$. Ans. $a^3 - b^3$.

16. Multiply $a^2 - a + 1$ by $a + 1$. Ans. $a^3 + 1$.

17. Required the product of $x^3 - ax^2 + a^2x - a^3$ and $x + a$. Ans. $x^4 - a^4$.

18. Required the product of $a^4 - a^3y + a^2y^2 - ay^3 + y^4$ and $a + y$. Ans. $a^5 + y^5$.

19. Required the product of $x^2 + y$ and $x^2 + y$.
Ans. $x^4 + 2x^2y + y^2$.

20. Required the product of $2ab - 3b^2$ and $3ab + 4b^2$.
Ans. $6a^2b^2 - ab^3 - 12b^4$.

21. Find the product of $x^2 + xy - y^2$ by $x - y$.
Ans. $x^3 - 2xy^2 + y^3$.

22. Find the product of $a^2 - 4a + 16$ by $a + 5$.
Ans. $a^3 + a^2 - 4a + 80$.

23. Find the product of $1 - a + a^2 - a^3$ by $1 + a$.
Ans. $1 - a^4$.

24. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.
Ans. $x^4 + x^2y^2 + y^4$.

25. Multiply $a - bx$ by $c - dx$.
Ans. $ac - (bc + ad)x + bdx^2$.

26. Multiply $3x^2 - 2xy - y^2$ by $2x - 4y$.
Ans. $6x^3 - 16x^2y + 6xy^2 + 4y^3$.

27. Multiply $x - y + z$ by $x + y - z$.
Ans. $x^2 - y^2 + 2yz - z^2$.

28. Multiply $27x^3 + 9x^2y + 3xy^2 + y^3$ by $3x - y$.
Ans. $81x^4 - y^4$.

29. Multiply $1 + x + x^4 + x^5$ by $1 - x + x^2 - x^3$.
Ans. $1 - x^8$.

30. Show that $a^n + y^n$ multiplied by $a^n + y^n$ is equal to $a^{2n} + 2a^ny^n + y^{2n}$.

65. The multiplication of polynomials may be *indicated* by inclosing each in a parenthesis, and writing them one after the other. When the operation indicated is actually performed, the expression is said to be *expanded*, or *developed*.

1. Expand $(a - b)(a - b)$. Ans. $a^2 - 2ab + b^2$.

2. Expand $(a + b)(c + d)$.
Ans. $ac + bc + ad + bd$.

3. Expand $(a + b)(a + b)(a + b)$.
Ans. $a^3 + 3a^2b + 3ab^2 + b^3$.

4. Develop $(a + b)(a + b)(a - b)$.
Ans. $a^3 + a^2b - ab^2 - b^3$.

5. Develop $(x^2 - xy + y^2)(x + y)$. Ans. $x^3 + y^3$.

How may the multiplication of polynomials be indicated? When is the expression said to be expanded, or developed?

6. Develop $(a+b+c)(a-b-c)$.
Ans. $a^2-b^2-2bc-c^2$.

7. Show that $(2x+3)(2x-3)(4x^2+9)=16x^4-81$.

8. Find the value of the expression $(a^n+b^n)(a-b)$.
Ans. $a^{n+1}+ab^n-a^nb-b^{n+1}$.

9. Find the value of the expression $(4a^m+6b^n)(a^m-b^3)$.
Ans. $4a^{2m}+6a^mb^n-4a^mb^3-6b^{n+3}$.

DIVISION.

66. DIVISION, in Algebra, is the process of finding how many times one quantity is contained in another;

Or, it is the process of finding one of two factors, when their product and the other factor are given.

The *Dividend* is the quantity to be divided.

The *Divisor* is the quantity by which we divide.

The *Quotient* is the result of the division.

Division is the converse of multiplication, the dividend corresponding to the product, and the divisor and quotient to the two factors.

67. *When dividend and divisor have* LIKE *signs, the quotient is* POSITIVE; *and when dividend and divisor have* DIFFERENT *signs, the quotient is* NEGATIVE.

For the quotient multiplied by the divisor must produce the dividend. Thus,

$$(+ab)\div(+b)=+a, \text{ for } (+a)\times(+b)=+ab;$$
$$(+ab)\div(-b)=-a, \text{ for } (-a)\times(-b)=+ab;$$
$$(-ab)\div(+b)=-a, \text{ for } (-a)\times(+b)=-ab;$$
$$(-ab)\div(-b)=+a, \text{ for } (+a)\times(-b)=-ab.$$

Hence, in division, as in multiplication,

Define Division. Dividend. Divisor. Quotient. When is the quotient positive? When negative?

Like signs produce $+$, *and unlike signs produce* $-$.

68. In division of algebraic quantities, there will be three cases: —

I. When both divisor and dividend are monomials.

II. When the divisor is a monomial and the dividend a polynomial.

III. When both divisor and dividend are polynomials.

CASE I.

69. When both divisor and dividend are monomials.

1. Let it be required to divide $14ab$ by $7a$.

OPERATION.

$$\frac{14ab}{7a} = 2b$$

Now, the quotient must be a quantity which, multiplied by $7a$, the divisor, will produce $14ab$, the dividend. Such a quantity is $2b$; which is obtained by rejecting from the dividend a factor equal to the divisor; or by dividing 14, the coefficient of the dividend, by 7, the coefficient of the divisor, and rejecting from the dividend the factor a, common to both.

2. Let it be required to divide a^5 by a^3.

OPERATION.

$$\frac{a^5}{a^3} = a^2$$

Since $a^5 = a\,a\,a\,a\,a$, and $a^3 = a\,a\,a$, it is evident that the quotient, or the quantity which, multiplied by the divisor a^3, will equal the dividend a^5, must be $a\,a$, or a^2. The exponent 2, in the quotient, which is the result of rejecting from the dividend a factor equal to the divisor, might have been obtained at once, by taking the difference of the exponents, 5 and 3. Hence,

The exponent of a letter in the quotient is equal to its exponent in the dividend, diminished by its exponent in the divisor.

In division what do like signs produce? Unlike signs? How many cases in division of algebraic quantities? Explain the first operation under Case I. The second. To what is the exponent of a letter in the quotient equal?

70. When the exponents of the same letter in the dividend and divisor are equal, the letter may be introduced into the quotient, without affecting the value of the expression, by writing the letter with the exponent 0.

For, let a^n be any power of the quantity a, then, dividing a^n by a^n, we have

$$\frac{a^n}{a^n} = a^{n-n} = a^0; \quad \text{but } \frac{a^n}{a^n} = 1.$$

Therefore (Ax. 7), $a^0 = 1$; and as a may have any value whatever,

Any quantity whose exponent is 0 *is equal to* 1.

Hence, by this notation the trace of a letter which has disappeared in an operation may be preserved, since the introduction of any factor whose value is unity will not affect the value of an expression.

Thus, $a^0 b$ has the same value as b alone.

71. When the exponent of any letter in the divisor is greater than it is in the dividend, the exponent of that letter in the quotient will be negative.

For, let it be required to divide a^2 by a^5, and we have (Art. 69),

$$\frac{a^2}{a^5} = a^{2-5} = a^{-3}.$$

Also, $\frac{a^2}{a^5} = \frac{1}{a^3}$; consequently, $a^{-3} = \frac{1}{a^3}$; that is,

Any quantity with a negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.

So, also, $$\frac{a^5}{a^3} = \frac{1}{a^{3-5}} = \frac{1}{a^{-2}}.$$

But, $\frac{a^5}{a^3} = a^2$; consequently, $\frac{1}{a^{-2}} = a^2$; hence,

---

What is the value of any quantity whose exponent is 0? When may a letter be introduced into the quotient without affecting the value expressed? What will be the character of the exponent in the quotient, when that of the divisor is greater than that of the dividend? To what is a quantity with a negative exponent equal?

Any factor may be transferred from the divisor to the dividend, or the reverse, by changing the sign of its exponent.

Note. As the signs + and — indicate opposite processes, qualities, or conditions (Art. 50), we should infer, from the relations of the signs themselves, that, if a *positive* exponent indicates the number of times a quantity is taken as a *factor*, a *negative* exponent must show the number of times it is used as a *divisor*. As the negative coefficient indicates subtraction, whether numerically possible or not (Art. 54, Exam. 2), so the negative exponent indicates division. (Art. 19, Note.) If the expression in which a negative exponent stands is already a divisor, then the quantity which it affects is a divisor of a divisor, and may be regarded as a factor of the dividend.

The relation of positive and negative exponents to each other, and to the exponent 0, is readily illustrated by such a series as the following, in which the exponents decrease regularly by one, to indicate a division by 3.

$$\begin{array}{ccccccc} 3^3, & 3^2, & 3^1, & 3^0, & 3^{-1}, & 3^{-2}, & 3^{-3} \\ 27, & 9, & 3, & 1, & \frac{1}{3}, & \frac{1}{9}, & \frac{1}{27} \end{array}$$

72. From the preceding examples and illustrations we have, for dividing one monomial by another, the following

RULE.

Divide the numerical coefficient of the dividend by that of the divisor, and to the result annex the literal factors of the dividend which are not found in the divisor.

Make the quotient positive, when the dividend and divisor have like signs, and negative when they have different signs.

Note. It is evident from the rule that one monomial cannot be exactly divided by another: —

1st. When the coefficient of the divisor is not exactly contained in that of the dividend.

2d. When the same letter has a greater exponent in the divisor than in the dividend.

3d. When the divisor contains one or more letters not found in the dividend.

How may any factor be transferred from the divisor to the dividend? Repeat the Rule. When is exact division of monomials impossible?

In each of these cases, the division is to be indicated by writing the divisor under the dividend, in the form of a fraction, or by the use of negative exponents.

EXAMPLES.

(1.)

$$\frac{9\,a\,b}{3\,a} = 3\,b$$

(2.)

$$\frac{12\,a^4\,b^5}{3\,a^2\,b^3} = 4\,a^2\,b^2$$

(3.)

$$\frac{-25\,x^3\,y\,z^2}{5\,x\,y\,z} = -5\,x^2\,z$$

(4.)

$$\frac{9\,m^2\,n^4\,x}{-m^2\,n} = -9\,n^3\,x$$

(5.)

$$\frac{21\,a\,d\,x^3}{-7\,a\,d\,x} = -3\,x^2$$

(6.)

$$\frac{-16\,a^3\,b^2}{-8\,a^5\,b} = 2\,a^{-2}\,b$$

7. Divide $16\,x^2$ by $8\,x$. Ans. $2\,x$.
8. Divide $7\,m\,x\,y$ by $x\,y$.
9. Divide $-x\,y$ by $x\,y$. Ans. $-x^0\,y^0$ or -1.
10. Divide $15\,a^2\,b^4$ by $5\,a\,b$. Ans. $3\,a\,b^3$.
11. Divide $-15\,a^2\,x^2$ by $5\,a\,x^2$.
12. Divide $10\,a\,n\,x\,y$ by $-2\,a\,y$. Ans. $-5\,n\,x$.
13. Divide $8\,x^8\,y^4$ by $-2\,x^7\,y$. Ans. $-4\,x\,y^3$.
14. Divide $-a^5$ by a^4. Ans. $-a$.
15. Divide $-16\,x^2\,y^3\,z^2$ by $-4\,x\,z$. Ans. $4\,x\,y^3\,z$.
16. Divide a^{m+n} by a^n. Ans. a^m.
17. Divide x^{m-n} by x^n. Ans. x^{m-2n}.
18. Divide $36\,a^7\,b^6\,c^2$ by $9\,a^5\,b^2\,c^4$. Ans. $4\,a^2\,b^4\,c^{-2}$.
19. Divide $(a+b)^5$ by $(a+b)^3$. Ans. $(a+b)^2$.
20. Divide $4\,(a-b)^2$ by $2\,(a-b)$. Ans. $2\,(a-b.)$
21. Divide $27\,a\,b^2\,(x+y)^3$ by $3\,b^3\,(x+y)^5$.
Ans. $9\,a\,b^{-1}\,(x+y)^{-2}$.

NOTE. In the last three examples, the expression in the parenthesis is to be considered as one quantity.

CASE II.

73. When the divisor is a monomial and the dividend a polynomial.

1. Divide $12\,a^3 b + 24\,a^3 c - 36\,a\,b$ by $12\,a$.

OPERATION.

$$12\,a)\ \underline{12\,a^3 b + 24\,a^3 c - 36\,a\,b}$$
$$a^2 b + 2\,a^2 c - 3\,b$$

Since the whole dividend must contain the divisor as many times as the latter is contained in the terms of the former, we divide $12 a^3 b$, $+ 24\,a^3 c$, and $-36\,a b$, respectively, by $12\,a$, and, connecting the partial quotients by their proper signs, have as the entire quotient, $a^2 b + 2\,a^2 c - 3\,b$.

RULE.

Divide each term of the dividend separately, and connect the results by their proper signs.

EXAMPLES.

(2.)

$$3\,a)\ \underline{6\,a\,x + 12\,a\,y}$$
$$2\,x + 4\,y$$

(3.)

$$a\,b)\ \underline{a^2 b + a\,b^2 - a\,b}$$
$$a + b - 1$$

(4.)

$$2\,x\,y)\ \underline{2\,x\,y - 6\,x\,y^2}$$
$$1 - 3\,y$$

(5.)

$$4\,a^2 c)\ \underline{12\,a^4 b\,c + 20\,a^3 b\,c - 8\,a^2 c^2}$$

(6.)

$$b\,x^2)\ \underline{5\,b\,x^2 + 10\,b\,x^3 - 12\,b^3 x^2}$$

7. Divide $9\,a^2 b^2 - 12\,a^3 c^3$ by $3\,a$.

Ans. $3\,a\,b^2 - 4\,a^2 c^3$.

8. Divide $12\,a^4 y^6 - 16\,a^5 y^5$ by $-4\,a^4 y^3$.

Ans. $-3\,y^3 + 4\,a\,y^2$.

Explain the operation. Repeat the Rule.

9. Divide $25\,b\,c^2 + 15\,x^2 - 5\,y$ by -5.
Ans. $-5\,b\,c^2 - 3\,x^2 + y$.

10. Divide $24\,a^3\,c^2\,x + 12\,a\,b^3\,c^4\,x$ by $-6\,a\,c^2\,x$.
Ans. $-4\,a^2 - 2\,b^3\,c^2$.

11. Divide $4\,a\,x\,y - 8\,a + 4\,a\,d$ by $4\,a$.
Ans. $x\,y - 2 + d$.

12. Divide $5\,(a - b)^2 + 10\,(a - b)$ by 5.
Ans. $(a - b)^2 + 2\,(a - b)$.

13. Divide $(x + y)^3 - (x + y)^2$ by $(x + y)$.
Ans. $(x + y)^2 - (x + y)$.

14. Divide $(a + c)^2 + 5\,(a + c)$ by $(a + c)$.
Ans. $(a + c) + 5$.

Note. When the parenthesis, as in the last answer, has neither coefficient nor exponent written, it may be dispensed with; thus, $(a + c) + 5$ may be written $a + c + 5$. If the parenthesis is preceded by the minus sign, all the signs of the inclosed terms should be changed. (Art. 55.)

15. Divide $a\,(x + y) - b\,(x + y)$ by $(x + y)$.
Ans. $a - b$.

16. Divide $3\,a\,(a + b) + (a + b)^2$ by $(a + b)$.
Ans. $4\,a + b$.

17. Divide $4\,(x + 3) - (x + 3)^2$ by $(x + 3)$.
Ans. $1 - x$.

18. Divide $15\,a^{-2}\,b^3\,x - 10\,a\,b\,x - 5\,a\,b^2\,x$ by $5\,a\,b^2\,x$.
Ans. $3\,a^{-3}\,b - 2\,b^{-1} - 1$.

CASE III.

74. When both divisor and dividend are polynomials.

1. Divide $a^3 + 5\,a^2\,x + 5\,a\,x^2 + x^3$ by $a + x$.

OPERATION.

$$\begin{array}{rllll}
a + x) & a^3 + 5a^2x & + 5ax^2 & + x^3 & (a^2 + 4ax + x^2 \\
 & a^3 + a^2x & & & \\
\cline{2-4}
 & \phantom{a^3 +{}} 4a^2x & + 5ax^2 & + x^3 & \\
 & \phantom{a^3 +{}} 4a^2x & + 4ax^2 & & \\
\cline{2-4}
 & & \phantom{+{}} ax^2 & + x^3 & \\
 & & \phantom{+{}} ax^2 & + x^3 & \\
\cline{3-4}
\end{array}$$

The terms of the divisor and dividend are arranged with reference to the decreasing powers of the letter a, so that the term having the highest exponent of a is placed first, that having the next highest immediately after, and so on; and, since the dividend is the product of the divisor and the quotient, the quotient will be arranged in the same order as the dividend.

Now, since the first term of the dividend, as arranged, must equal the product of the first term of the divisor by the first term of the quotient, or that having the highest power of a, we divide a^3 by a, and obtain a^2 for the first term of the quotient.

The product of the whole divisor by this term, or $a^3 + a^2x$, subtracted from the whole dividend, leaves $4a^2x + 5ax^2 + x^3$. This remainder may be regarded as a *new* dividend, produced by multiplying each term of the divisor by each of the remaining terms of the quotient.

The first term of this new dividend must have been produced by the first term of the divisor multiplied by the second term of the quotient, we therefore divide $4a^2x$ by a, and obtain $4ax$ as the second term of the quotient.

The product of the whole divisor by this term, or $4a^2x + 4ax^2$, subtracted from the second dividend, leaves $ax^2 + x^3$, for a third dividend. Dividing ax^2, the first term of this dividend, by a, the first term of the divisor, we obtain x^2, as the third term of the quotient; and multiplying the whole divisor by x^2, obtain $ax^2 + x^3$, which subtracted from the last dividend leaves no remainder.

Since, at each step in the operation, we divide the term containing the highest exponent of some letter in the div-

---

Explain the operation.

idend, by the term containing the highest exponent of the same letter in the divisor,

The terms of the divisor and dividend should always be arranged in the order of the powers of some letter common to both.

From what precedes, we deduce, for the division of one polynomial by another, the following

RULE.

Arrange both dividend and divisor according to the powers of some common letter.

Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient; by which multiply the whole divisor, and subtract the product from the dividend.

Regard the remainder as a new dividend, find the next term of the quotient, in the same manner as before, and proceed with it as with the first quotient, and so on.

NOTE 1. The divisor is sometimes placed on the right of the dividend, that it may be the more readily multiplied by the several terms of the quotient, as they are found.

NOTE 2. It will not be necessary to bring down any more terms of the dividend to form the remainder at each successive subtraction than are required by the quantity to be subtracted.

NOTE 3. When the first term of an arranged dividend is not divisible by the first term of the arranged divisor, exact division is impossible; and when it is thus found that any remainder is not divisible by the divisor, that remainder, with the divisor under it, in the form of a fraction, should be written after the quotient found.

NOTE 4. The work in division may be proved, as in Arithmetic, by multiplication.

How should the terms of the divisor and dividend be arranged? Repeat the Rule. Where is the divisor sometimes placed? What is said in Note 2? When is exact division impossible? What is to be done with the last remainder? How may division be proved?

EXAMPLES.

2. Divide $a^2+2ab+b^2$ by $a+b$.

OPERATION.

$$\begin{array}{l|l} a^2+2ab+b^2 & a+b \\ \underline{a^2+ab} & \overline{a+b} \\ ab+b^2 & \\ \underline{ab+b^2} & \end{array}$$

$$\text{Proof}\left\{\begin{array}{l} a+b \\ \underline{a+b} \\ a^2+ab \\ \underline{ab+b^2} \\ a^2+2ab+b^2 \end{array}\right.$$

3. Divide a^3-b^3 by $a+b$.

OPERATION.

$$\begin{array}{l} a+b)\,a^3-b^3\,(a^2-ab+b^2-\frac{2b^3}{a+b} \\ \underline{a^3+a^2b} \\ -a^2b-b^3 \\ \underline{-a^2b-ab^2} \\ ab^2-b^3 \\ \underline{ab^2+b^3} \\ -2b^3 \end{array}$$

4. Divide a^2-b^2 by $a+b$. Ans. $a-b$.

5. Divide $a^3+2a^2b+2ab^2+b^3$ by a^2+ab+b^2. Ans. $a+b$.

6. Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$. Ans. $a^2-2ab+b^2$.

7. Divide a^3-1 by $a-1$. Ans. a^2+a+1.

8. Divide $8a^3-4a^2b-6ab^2+3b^3$ by $2a-b$. Ans. $4a^2-3b^2$.

9. Divide x^4+4x+3 by x^2-2x+3. Ans. x^2+2x+1.

10. Divide a^3-x^3 by a^2+ax+x^2. Ans. $a-x$.

11. Divide $a^4+4a^2b^2+16b^4$ by $a^2-2ab+4b^2$. Ans. $a^2+2ab+4b^2$.

12. Divide $a^3 - x^3$ by $a - x$. Ans. $a^2 + ax + x^2$.

13. Divide $a^3 + x^3$ by $a + x$. Ans. $a^2 - ax + x^2$.

14. Divide $x^3 - 5x^2 - 46x - 40$ by $x + 4$.
Ans. $x^2 - 9x - 10$.

15. Divide $x^7 - 1$ by $x - 1$.
Ans. $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

16. Divide $a^5 + x^5$ by $a + x$.
Ans. $a^4 - a^3x + a^2x^2 - ax^3 + x^4$.

17. Divide $21a^5 - 21b^5$ by $7a - 7b$.
Ans. $3a^4 + 3a^3b + 3a^2b^2 + 3ab^3 + 3b^4$.

18. Divide $2a^4 + 2a^3b + 5a^2b^2 - 6ab^3 + 4b^4$ by $2a^2 - 2ab + b^2$. Ans. $a^2 + 2ab + 4b^2$.

19. Divide $2a^{m+1} - 2a^{n+1} - a^{m+n} + a^{2n}$ by $2a - a^n$.
Ans. $a^m - a^n$.

20. Divide $a^4 - 3x^4$ by $a + x$.
Ans. $a^3 - a^2x + ax^2 - x^3 - \frac{2x^4}{a+x}$.

21. Divide $a^5b^0 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - a^0b^5$ by $a^2 - 2ab + b^2$.
Ans. $a^3 - 3a^2b + 3ab^2 - b^3$.

THEOREMS.

75. A FORMULA, is an algebraic expression of a general rule.

The following theorems give rise to formulas, useful in abridging algebraic operations.

THEOREM I.

76. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

Define a Formula. What is Theorem I.?

For, let a represent one of the quantities, and b the other; then,

$$(a+b)^2 = (a+b) \times (a+b) = a^2 + 2ab + b^2.$$

Hence, the theorem is true.

EXAMPLES.

1. Find the square of $3a + x^2$. From the formula, we have

$$(3a + x^2)^2 = 9a^2 + 6ax^2 + x^4.$$

2. Square $2x + y$. Ans. $4x^2 + 4xy + y^2$.

3. Square $6a^2 + 2a^2b$.
Ans. $36a^4 + 24a^4b + 4a^4b^2$.

4. Square $a^3b^2 + 3a^2b^3c^4$.
Ans. $a^6b^4 + 6a^5b^5c^4 + 9a^4b^6c^8$.

THEOREM II.

77. ***The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.***

For, let a represent one of the quantities, and b the other; then,

$$(a-b)^2 = (a-b) \times (a-b) = a^2 - 2ab + b^2,$$

which proves the theorem.

EXAMPLES.

1. Find the square of $3x - a$. We have

$$(3x - a)^2 = 9x^2 - 6ax + a^2.$$

2. Square $5c - 1$. Ans. $25c^2 - 10c + 1$.

3. Square $a^2 - b^2$. Ans. $a^4 - 2a^2b^2 + b^4$.

4. What is the square of $5a^2b^2 - 10a^2b^3$?
Ans. $25a^4b^4 - 100a^4b^5 + 100a^4b^6$.

Give its demonstration. What is Theorem II.? Give its demonstration.

THEOREM III.

78. *The product of the sum and difference of two quantities is equal to the difference of their squares.*

For, let a represent one of the quantities, and b the other; then,

$$(a+b) \times (a-b) = a^2 - b^2,$$

which agrees with the theorem.

EXAMPLES.

1. Find the product of $3a + 2b$ by $3a - 2b$.
Ans. $9a^2 - 4b^2$.

2. Required the product of $a + y$ by $a - y$.
Ans. $a^2 - y^2$.

3. Multiply $5a + b$ by $5a - b$. Ans. $25a^2 - b^2$.

4. Multiply $9x + 1$ by $9x - 1$. Ans. $81x^2 - 1$.

5. What is the product of $3a^2c + 10ab^3$ by $3a^2c - 10ab^3$? Ans. $9a^4c^2 - 100a^2b^6$.

6. Multiply $3x^2y + 12xy^3$ by $3x^2y - 12xy^3$.
Ans. $9x^4y^2 - 144x^2y^6$.

MISCELLANEOUS EXAMPLES.

1. Required the square of $m - n$.
Ans. $m^2 - 2mn + n^2$.

2. Multiply $3a - 2$ by $3a - 2$.
Ans. $9a^2 - 12a + 4$.

3. Expand $(9ab + 2b^2)^2$.
Ans. $81a^2b^2 + 36ab^3 + 4b^4$.

4. What is the product of $a - d$ by $a - d$?
Ans. $a^2 - 2ad + d^2$.

5. Expand $(2 - x^m)(2 - x^m)$. Ans. $4 - 4x^m + x^{2m}$.

What is Theorem III.? Give its demonstration.

6. Multiply $a^3 + 1$ by $a^3 - 1$. Ans. $a^6 - 1$.

7. Expand $(a^2 - b^2)(a^2 + b^2)$. Ans. $a^4 - b^4$.

8. Square $1 - 3c^2$. Ans. $1 - 6c^2 + 9c^4$.

9. Expand $(2 + \overline{a - b})(2 - \overline{a - b})$. Ans. $4 - (a - b)^2$.

10. Expand $2(a + b)(a - b)$. Ans. $2a^2 - 2b^2$.

11. Expand $3^3(x^2 - a^2)^2$. Ans. $27x^4 - 54a^2x^2 + 27a^4$.

12. Expand $(1 - 4a)(1 - 4a)$. Ans. $1 - 8a + 16a^2$.

13. Expand $(3m + 4n)(3m - 4n)$. Ans. $9m^2 - 16n^2$.

14. Expand $(3a - 4x)(3a + 4x)$.

15. Expand $(2a + 3x)(2a + 3x)$. Ans. $4a^2 + 12ax + 9x^2$.

16. Expand $(2ac - 3bc)(2ac - 3bc)$. Ans. $4a^2c^2 - 12abc^2 + 9b^2c^2$.

17. Expand $(3 - \overline{a + b})(3 - \overline{a + b})$. Ans. $9 - 6(a + b) + (a + b)^2$.

18. Expand $(5a^2b^2 + 7ab)(5a^2b^2 - 7ab)$. Ans. $25a^4b^4 - 49a^2b^2$.

19. Expand $(x + a)(x - a)(x^2 - a^2)$. Ans. $x^4 - 2a^2x^2 + a^4$.

Note. In expanding the first two factors, apply Theorem III.; and the Theorem II.

20. Expand $(x + 2)(x - 2)(x - 3)(x + 3)$. Ans. $x^4 - 13x^2 + 36$.

21. Expand $(2x + 3)(2x - 3)(4x^2 + 9)$. Ans. $16x^4 - 81$.

22. Find the value of $(x^2 - 1)(x^2 + 1)(x^4 - 1)$. Ans. $x^8 - 2x^4 + 1$.

FACTORING.

79. FACTORING is the process of resolving a quantity into its factors.

80. The FACTORS of a quantity are such integral quantities as, when multiplied together, will produce the given quantity.

81. A PRIME QUANTITY is one that cannot be divided, without a remainder, by any integral quantity different from itself, or unity. Thus, a, b, and $a + c$ are prime quantities.

82. A COMPOSITE QUANTITY is one that can be divided, without a remainder, by some integral quantity other than itself, or unity. Thus, a^2, $a b$, and $a b + a c$ are composite quantities.

83. Quantities are said to be *prime to each other*, when they have no common factor greater than unity.

84. One quantity is said to be divisible by another when the latter will divide the former without a remainder. Thus, $a b$ is divisible by either a or b.

A composite quantity is divisible by any of its factors, and a prime quantity only by itself and unity.

85. *The difference of any two equal powers of two quantities is always divisible by the difference of the quantities.*

For, let a and b represent any two quantities, a being greater than b; then,

$$(a^2 - b^2) \div (a - b) = a + b,$$
$$(a^3 - b^3) \div (a - b) = a^2 + a b + b^2,$$
$$(a^4 - b^4) \div (a - b) = a^3 + a^2 b + a b^2 + b^3,$$

and so on.

Define Factoring. Factors. Prime Quantity. Composite Quantity. When are quantities prime to each other? When is one quantity divisible by another? By what is the difference of any two equal powers of two quantities divisible?

Also, let b or a become 1; then, $(a^2-1)\div(a-1)=a+1$; $(a^3-1)\div(a-1)=a^2+a+1$; $(1-b^2)\div(1-b)=1+b$, etc.

86. *The difference of two equal even powers of two quantities is always divisible by the sum of the quantities.*

For, $(a^2-b^2)\div(a+b)=a-b$,

$(a^4-b^4)\div(a+b)=a^3-a^2b+ab^2-b^3$,

$(a^6-b^6)\div(a+b)=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$,

and so on.

Also, $(a^2-1)\div(a+1)=a-1$; $(a^4-1)\div(a+1)=a^3-a^2+a-1$; $(1-b^2)\div(1+b)=1-b$, etc.

87. *The sum of two equal odd powers of two quantities is always divisible by the sum of the quantities.*

For,

$(a^3+b^3)\div(a+b)=a^2-ab+b^2$,

$(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4$,

$(a^7+b^7)\div(a+b)=a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6$,

and so on.

Also, $(a^3+1)\div(a+1)=a^2-a+1$; $(a^5+1)\div(a+1)=a^4-a^3+a^2-a+1$; $(1+b^3)\div(1+b)=1-b+b^2$, etc.

CASE I.

88. To resolve monomials into their prime factors.

1. Find the prime factors of $12a^2b$.

OPERATION.

$$\begin{array}{l} 12=2\times2\times3 \\ a^2=a\times a \\ b\;=b \\ \hline 12a^2b=2\times2\times3\,aab \end{array}$$

Since the composite factor 12 is the product of the prime factors 2, 2, and 3, the composite factor a^2, of a and a, and b is prime, the prime factors of $12a^2b$ are 2, 2, 3, a, a, and b, or $2\times2\times3\,aab$. Hence the following

---

By what is the difference of two even powers of the same degree divisible? By what the sum of two odd powers of the same degree? Explain the operation under Case I.

RULE.

To the prime factors of the numerical coefficient annex each letter written as many times as there are units in its exponent.

EXAMPLES.

2. Find the prime factors of $8ab^3$.
Ans. $2 \times 2 \times 2abbb$.

3. Resolve $21m^3n^2x$ into its prime factors.

4. Factor $49a^2bx^2y^3$. Ans. $7 \times 7aabxxyyy$.

5. Find the prime factors of $56a^3b^4c^2x^3y$.

6. Factor $81b^3c^2dx^3$.
Ans. $3 \times 3 \times 3 \times 3bbbccdxxx$.

CASE II.

89. To resolve a polynomial into two factors, when one of them is a monomial.

1. Find the factors of $ac + bc$.

OPERATION.

$$(ac + bc) \div c = a + b$$
$$ac + bc = c(a + b)$$

Since c is a factor common to all the terms, we divide $ac + bc$ by c, and obtain for the other factor, $a + b$; whence, $ac + bc = c(a + b)$.

RULE.

Divide the polynomial by the greatest monomial factor common to all its terms, and to the quotient, inclosed in a parenthesis, prefix the divisor as a coefficient.

Repeat the Rule. Explain the operation under Case II. Repeat the Rule.

EXAMPLES.

2. Factor $a b^2 + a b c$. Ans. $a b (b + c)$.

3. Resolve $a + a y$ into its factors. Ans. $a (1 + y)$.

4. Resolve $a x + x$ into its factors.

5. Find the factors of $6 x^4 + 3 x^3 - 9 x$.
Ans. $3 x (2 x^3 + x^2 - 3)$.

6. Resolve $14 b c^4 x - 21 b^2 c^3 x + 7 b c^2 x$ into factors.

7. Factor $16 a^3 - 12 a b + 4 a c$.
Ans. $4 a (4 a^2 - 3 b + c)$.

8. Find the factors of $77 a^2 x - 11 a^2 y + 22 a c$.
Ans. $11 a (7 a x - a y + 2 c)$.

9. Factor $21 x^3 y^2 + 14 x^2 y + 21 x y$.
Ans. $7 x y (3 x^2 y + 2 x + 3)$.

10. Resolve $14 a^4 b^4 x^9 y - 6 a x^5 y^5 + 10 a x^3 y$ into factors.
Ans. $2 a x^3 y (7 a^3 b^4 x^6 - 3 x^2 y^4 + 5)$.

CASE III.

90. **To resolve a trinomial into two equal binomial factors.**

Any trinomial can be resolved into two equal binomial factors, when two of the terms are squares and positive, and the other term is twice the product of their square roots.

1. Find the factors of $a^2 + 2 a b + b^2$.

OPERATION.

$$\begin{aligned} a^2 &= \text{the square of } a, \\ b^2 &= \text{the square of } b, \\ 2 a b &= \text{twice } a \times b. \\ a^2 + 2ab + b^2 &= (a + b)(a + b). \end{aligned}$$

Since a^2 is the square of a, b^2 is the square of b, and $2 a b$ is twice the product of a and b, and is positive, we have, by Theorem I., Art. 76, for the two factors, $(a + b)(a + b)$. But had the middle term been negative, then, by Theorem II., Art. 77, the factors would have been $(a - b)(a - b)$. Hence the following

---

Explain the operation under Case III.

RULE.

Take for each of the required factors the sum or difference of the square roots of the square terms, according as the other term is positive or negative.

NOTE. Some trinomials may be resolved into *unequal* binomial factors, thus, $x^2 - 5x + 6 = (x - 3)(x - 2)$, $2a^2 + ab - 6b^2 = (2a - 3b)(a + 2b)$; but no simple general principle can be applied to all such cases.

EXAMPLES.

2. Factor $a^2 - 2ab + b^2$. Ans. $(a - b)(a - b)$.
3. Factor $4a^2 + 12ab + 9b^2$. Ans. $(2a + 3b)(2a + 3b)$.
4. Resolve $4a^2 - 12ab + 9b^2$ into its factors.
5. Factor $a^2 - 4ab^2 + 4b^4$. Ans. $(a - 2b^2)(a - 2b^2)$.
6. Factor $x^2 - 2x + 1$. Ans. $(x - 1)(x - 1)$.
7. Required the two binomial factors of $1 + 2x^2 + x^4$.
8. Resolve $4x^2 + 4xy + y^2$ into its factors. Ans. $(2x + y)(2x + y)$.
9. Factor $25m^4 + 10m^2n + n^2$. Ans. $(5m^2 + n)(5m^2 + n)$.

91. To resolve a binomial into two binomial factors.

Any binomial can be resolved into two binomial factors when its terms represent the difference between two squares.

1. Find the binomial factors of $a^2 - b^2$.

Repeat the Rule.

OPERATION.

$$a^2 = \text{the square of } a,$$
$$b^2 = \text{the square of } b.$$
$$a^2 - b^2 = (a + b)(a - b).$$

Since a^2 is the square of a, and b^2 is the square of b, we have, by Theorem III, Art. 78, for the required factors, the sum and difference of a and b, or $(a + b)(a - b)$. Hence the

RULE.

Take for one of the factors the sum, and for the other the difference, of the square roots of the given terms.

2. Factor $a^2 - c^2$. Ans. $(a + c)(a - c)$.
3. Find the factors of $x^2 - y^2$.
4. Factor $4x^2 - y^2$. Ans. $(2x + y)(2x - y)$.
5. Factor $9a^2 - 4b^2$. Ans. $(3a + 2b)(3a - 2b)$.
6. Find the binomial factors of $64a^2b^2 - 16c^2d^2$.
7. Factor $1 - 81x^2$. Ans. $(1 + 9x)(1 - 9x)$.
8. Factor $c^6 - a^4y^2$. Ans. $(c^3 + a^2y)(c^3 - a^2y)$.
9. Factor $a^4 - b^4$.

NOTE. $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)$.

10. Factor $1 - c^4$. Ans. $(1 + c^2)(1 + c)(1 - c)$.
11. Factor $16y^8 - 1$.
 Ans. $(4y^4 + 1)(2y^2 + 1)(2y^2 - 1)$.
12. Factor $a^8 - c^8$.
 Ans. $(a^4 + c^4)(a^2 + c^2)(a + c)(a - c)$.
13. Factor $x^8 - y^4$. Ans. $(x^4 + y^2)(x^2 + y)(x^2 - y)$.

92. Any binomial which consists of the *difference* of any two equal powers, or the *sum* of any two equal *odd* powers, may be factored by aid of Articles 85, 86, and

---

Explain the operation. Repeat the Rule. When may a binomial be factored upon the principles contained in Articles 85, 86, and 87?

87; for the quotient and divisor are factors of the dividend.

1. Factor $a^3 - b^3$.

OPERATION.

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2$$

Since, by Art. 85, $a - b$ is a factor of $a^3 - b^3$, we divide the latter by the former, and obtain as another factor, $a^2 + ab + b^2$; and thus have $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

2. Factor $a^3 + b^3$. Ans. $(a + b)(a^2 - ab + b^2)$.

3. Factor $m^4 - n^4$.

Ans. $(m - n)(m^3 + m^2 n + m n^2 + n^3)$,
or $(m + n)(m^3 - m^2 n + m n^2 - n^3)$.

4. Factor $1 - x^4$.

Ans. $(1 - x)(1 + x + x^2 + x^3)$,
or $(1 + x)(1 - x + x^2 - x^3)$.

NOTE. The second factor, in either answer of the last two examples, may be again resolved into factors, so that either set of answers will reduce to the same form as those of Examples 9 and 10 in the last Article.

5. Factor $8x^3 - y^3$.

Ans. $(2x - y)(4x^2 + 2xy + y^2)$.

6. Factor $8x^3 + 1$. Ans. $(2x + 1)(4x^2 - 2x + 1)$.

7. Factor $a^5 + b^5$.

Ans. $(a + b)(a^4 - a^3 b + a^2 b^2 - a b^3 + b^4)$.

8. Resolve into factors $a^6 - b^6$.

Ans. $(a^3 + b^3)(a^3 - b^3) = (a^3 + b^3)(a - b)(a^2 + ab + b^2)$
$= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2)$
$= (a^2 - b^2)(a^4 + a^2 b^2 + b^4)$.

NOTE. The factor $a^2 - b^2$ is found by taking the product of the factors $a + b$ and $a - b$; and either using it as a divisor, or multiplying the remaining factors together, gives the factor $a^4 + a^2 b^2 + b^4$. By making the factors $a + b$ and $a - b$ divisors, other factors can be obtained.

Explain the operation.

93. In factoring many polynomials, much must depend upon the skill of the learner, since specific directions cannot well be given to meet every case.

Sometimes a portion only of a polynomial can be factored, as when the terms do not all have a common factor.

1. Factor $a^2c + 2abc + b^2c$.

Ans. $c(a+b)(a+b)$.

Note. $a^2c + 2abc + b^2c = c(a^2 + 2ab + b^2)$, and $a^2 + 2ab + b^2 = (a + b(a + b)$.

2. Factor $ab + ad + cx + cy$.

Ans. $a(b+d) + c(x+y)$.

3. Factor $2y^3 + 3x^2y - 9x^2$.

Ans. $2y^3 + 3x^2(y-3)$.

4. Factor $6x^3 + 12x^2y + 6xy^2$.

Ans. $6x(x+y)(x+y)$.

5. Factor $ab + ay + bx + xy$.

Ans. $(a+x)(b+y)$.

Note. $ab + ay + bx + xy = a(b+y) + x(b+y) = (a+x)(b+y)$.

6. Factor $ac - bd + bc - ad$.

Ans. $(a+b)(c-d)$.

7. Factor $6ax - 2by + 3bx - 4ay$.

Ans. $(2a+b)(3x-2y)$.

Note. The product of two binomial factors reduces to a trinomial whenever two of the partial products are similar. (Art. 90, Note.)

GREATEST COMMON DIVISOR.

94. A Divisor or Measure of a quantity is any quantity that will divide it without a remainder.

What is said of factoring polynomials? When can a polynomial be only partially factored? Define a Divisor or Measure.

95. A COMMON DIVISOR or MEASURE of two or more quantities is a quantity that will divide each of them without a remainder.

Hence, any factor common to two or more quantities is a common divisor of those quantities.

96. The GREATEST COMMON DIVISOR of two or more quantities is the greatest quantity that will divide each of them without a remainder.

Hence, the greatest common divisor of two or more quantities is composed of all the factors common to those quantities.

When quantities are prime to each other (Art. 83), they have no common measure greater than unity.

97.* The greatest common divisor of two quantities is also the greatest common divisor of the least quantity and their remainder after division.

For, let a and b be two quantities, of which b is the least.

Suppose, now, that b is not contained in a an exact number of times, but m times, with a remainder, r. Then, since the dividend is equal to the product of the divisor by the quotient, plus the remainder, we have

$$a = mb + r.$$

Also, since the remainder is equal to the dividend minus the product of the divisor by the quotient,

$$r = a - mb.$$

Now, any quantity that will exactly divide b will exactly divide m times b, or mb; and any quantity that will exactly divide b and r will exactly divide mb and r, and consequently will exactly divide

Define Common Divisor. Greatest Common Divisor. Prove that the greatest common divisor of two quantities is the same as the greatest common divisor of the least, and their remainder after division.

---

* Beginners, at the option of the teacher, may omit this Article.

their sum, $mb + r$, or its equal, a. Hence, any quantity that is a common divisor of b and r is also a common divisor of a and b.

Again, any quantity that will exactly divide a and b will exactly divide a and mb, and consequently will exactly divide their difference, $a - mb$, or its equal, r. Therefore, any common divisor of a and b must also be a common divisor of b and r.

But the converse of this has already been proved; consequently, the common divisors of a and b, and of b and r, must be identical, and the *greatest* common divisor of a and b must be also the *greatest* common divisor of b and r; which was to be proved.

NOTE. It will be seen that the greatest common divisor of a and b is common to the four quantities a, b, mb, and r, that is, to the dividend, divisor, product of the divisor by the quotient, and remainder; but it is not necessarily found in the quotient, m. The divisor, b, and remainder, r, most nearly approach the common divisor, as they are smaller than either of the others which contain it, or they contain a less number of other factors. Moreover, the greatest common divisor of a and b is not, necessarily, the *greatest* common divisor of any other two of the four quantities involved, when taken by themselves. 24 and 9 are convenient numbers to be used for a and b in illustrating these principles.

CASE I.

98. To find the greatest common divisor of monomials.

1. Find the greatest common divisor of $4a^2b^3c$ and $6a^3bc^2d$.

OPERATION.

$$\begin{array}{ll} 4a^2b^3c & = 2^2 \times a^2 \times b^3 \times c \\ 6a^3bc^2d & = 3 \times 2 \times a^3 \times b \times c^2 \times d \\ \hline 2a^2bc & = 2 \times a^2 \times b \times c \end{array}$$

Resolving the quantities into factors, we find that 2, a^2, b, and c are the only common factors; and since the product of these, or $2a^2bc$, is composed of all the factors common to the quantities (Art. 96), it is their greatest common divisor. Hence the

---

Explain the operation.

RULE.

Resolve the quantities into their prime factors, and the product of all the factors common to the several quantities will be the greatest common divisor.

NOTE. Any letter forming a part of the common divisor will take the lowest exponent with which it occurs in either of the original quantities.

EXAMPLES.

2. Find the greatest common divisor of $15\,a^3 b^2 c^2$ and $12\,a^3 b\,c^3 x$. Ans. $3\,a^3 b\,c^2$.

3. Find the greatest common divisor of $8\,x^6 y^5$, $4\,x^3 y^4$, and $10\,x^5 y^4$. Ans. $2\,x^3 y^4$.

4. Find the greatest common divisor of $5\,a^3 b\,c^3 d^2$, $10\,a\,b^3 c^2 d^3$, and $15\,a^2 b^2 c\,d^2$.

5. Find the greatest common divisor of $9\,a^3 b^2 m^5 n$, $12\,a^3 b^3 m^5 n^2$, and $15\,a^3 b^2 m^5 n^3$. Ans. $3\,a^3 b^2 m^5 n$.

l

CASE II.

99.* To find the greatest common divisor of polynomials.

1. Find the greatest common divisor of $x^2 + 2x + 1$ and $x^3 + 2x^2 + 2x + 1$.

OPERATION.

$$\begin{array}{lllll}
x^2+2x+1) & x^3+2x^2+2x+1\,(x & & & \\
 & \underline{x^3+2x^2+\ \ x\qquad} & & & \\
 & & x+1) & x^2+2x+1\,(x+1 & \\
 & & & \underline{x^2+\ \ x} & \\
 & & & \qquad x+1 & \\
 & & & \qquad \underline{x+1} &
\end{array}$$

Repeat the Rule. Explain the operation.

---

* Beginners, at the option of the teacher, may omit this Article.

It is evident that, if $x^2 + 2x + 1$ will exactly divide $x^3 + 2x^2 + 2x + 1$, it will be the greatest common divisor, since no quantity can have a divisor greater than itself. But we find that the latter is not divisible by the former, there being a remainder, $x + 1$. Now, we know that the greatest common divisor cannot be greater than this remainder; for the greatest common divisor of two quantities must be a divisor of their remainder after division (Art. 97). We, therefore, divide the divisor by the remainder, which it exactly divides. As $x + 1$ is the greatest common divisor of the remainder and divisor, it must also be that of the divisor and dividend (Art. 97); consequently it is the greatest common divisor required.

Hence, for finding the greatest common divisor of two polynomials, we have the following

RULE.

Divide the greater quantity by the less, and if there is no remainder, the less quantity will be the divisor required.

If there is a remainder, divide the divisor by it, and continue thus to make the preceding divisor the dividend and the remainder the divisor, until a divisor is obtained which leaves no remainder; the last divisor will be the greatest common divisor.

NOTE 1. When the two quantities are expressions of the *same degree*, it is immaterial which is made the divisor.

NOTE 2. If both polynomials have a common monomial factor, it may be suppressed during the operation; but it must finally be restored as a factor of the common divisor.

NOTE 3. If either polynomial has a monomial factor not common to the two, it may be suppressed, since such a factor can form no part of the greatest common divisor.

NOTE 4. If the leading term of any dividend is not divisible by the first term of the divisor, each term of the dividend may be multiplied by any quantity, not a factor of all the terms of the divisor, which will

---

Repeat the Rule. What is Note 1? Note 2? Note 3? Note 4?

render it divisible; since the factor thus introduced, not being a common factor, cannot affect the common measure.

NOTE 5. If any of the divisors, in the course of the operation, become negative, they may have their signs changed, since a change of all the signs in either divisor or dividend does not affect the question of divisibility.

NOTE 6. When the greatest common divisor of *more* than two quantities is required, find the greatest common divisor of two of them, and then of that common divisor and one of the other quantities, and so on, for all the given quantities. The last common divisor will be the greatest common divisor required.

EXAMPLES.

2. Find the greatest common divisor of $2a^2x-2b^2x$ and $4a^3x+4b^3x$.

$$\begin{array}{l}
2a^2x-2b^2x)\,4a^3x+4b^3x \\
\quad a^2-b^2)\;a^3+b^3\,(a \\
\qquad\qquad\;\; \underline{a^3-ab^2} \\
\qquad\qquad\qquad\quad ab^2+b^3 \\
\qquad\qquad\qquad\quad a+b)\,a^2-b^2\,(a-b \\
\qquad\qquad\qquad\qquad\qquad \underline{a^2+ab} \\
\qquad\qquad\qquad\qquad\qquad -ab-b^2 \\
\qquad\qquad\qquad\qquad\qquad \underline{-ab-b^2}
\end{array}$$

Ans. $2x(a+b)$.

We suppress the factor $2x$ in the first quantity, and $4x$ in the second, and find a common factor, $2x$, in both, which we reserve as a factor in the greatest common divisor (Note 2). The quantities now become a^2-b^2 and a^3+b^3. In the first remainder we suppress the factor b^2 (Note 3), and it becomes $a+b$. The product of the last divisor, $a+b$, by the common factor, $2x$, gives $2x(a+b)$ as the greatest common divisor required.

3. Find the greatest common divisor of $3x^2-2x-1$ and $4x^3-2x^2-3x+1$.

What is Note 5? Note 6? Explain the operation.

$$\begin{array}{r|l}
4x^3-2x^2-3x+1 & 3x^2-2x-1 \\
3 & (4x \\
\hline
12x^3-6x^2-9x+3 & \\
\underline{12x^3-8x^2-4x} & \\
2x^2-5x+3)\ 3x^2-2x-1\ (3 & \\
2 & \\
\hline
6x^2-\ \ 4x-2 & \\
\underline{6x^2-15x+9} & \\
11x-11 & \\
x-1)\ 2x^2-5x+3\ (2x-3 & \\
\underline{2x^2-2x} & \\
-3x+3 & \\
\underline{-3x+3} &
\end{array}$$

Ans. $x-1$.

We multiply by 3 in the first instance, and by 2 in the second, to make the division possible (Note 4), and suppress in the second remainder the factor 11 (Note 3).

The first divisor is written at the right, in order to economize space.

4. Find the greatest common divisor of $3x^3-24x-9$ and $2x^3-16x-6$. Ans. x^3-8x-3.

5. Find the greatest common divisor of $4a^2-4ax-15x^2$ and $6a^2+7ax-3x^2$. Ans. $2a+3x$.

6. Find the greatest common divisor of $2a^2+ab-b^2$, $a^2-2ab-3b^2$, and $3ac+3bc$. Ans. $a+b$.

7. Find the greatest common divisor of $2x^4-7x^3+5x^2$ and x^3+3x^2-4x. Ans. x^2-x.

100. The greatest common divisor of polynomials may often be most readily obtained by factoring, after the manner of monomials. (Art. 98.)

Explain the operation. In what way may the greatest common divisor of polynomials often be most readily found?

1. Find the greatest common divisor of $3a^2 - 3b^2$ and $3a^2 + 6ab + 3b^2$. Ans. $3(a + b)$.

NOTE. $3a^2 - 3b^2 = 3(a^2 - b^2) = 3(a + b)(a - b)$, and
$3a^2 + 6ab + 3b^2 = 3(a^2 + 2ab + b^2) = 3(a + b)(a + b)$.

2. Required the greatest common divisor of $ab + b^2$ and $ac^2 + bc^2$. Ans. $a + b$.

3. What is the greatest common divisor of $a^2 - 2a$ and $ab - 2b$? Ans. $a - 2$.

4. Required the greatest common divisor of $a^5 - a^3b^2$ and $a^4 - b^4$. Ans. $a^2 - b^2$.

5. Find the greatest common divisor of $ab + am + bn + mn$ and $b^2n - m^2n$. Ans. $b + m$.

6. Find the greatest common divisor of $a^2 + 2ab + b^2$ and $a^3 - ab^2$. Ans. $a + b$.

7. Required the greatest common divisor of $3x^2 - 3y^2$, $3x^2 + 6xy + 3y^2$, and $3abx + 3aby$. Ans. $3(x + y)$.

LEAST COMMON MULTIPLE.

101. A MULTIPLE of a quantity is any quantity that can be divided by it without a remainder.

Hence, a multiple of a quantity must contain all the prime factors of that quantity.

102. A COMMON MULTIPLE of two or more quantities is one that can be divided by each of them without a remainder.

Hence, a common multiple of two or more quantities must contain all the prime factors of each of the quantities.

103. The LEAST COMMON MULTIPLE of two or more quantities is the least quantity that can be divided by each of them without a remainder.

Define a Multiple. Define a Common Multiple of two or more quantities. Least Common Multiple.

Hence, *the least common multiple of two or more quantities must be the product of all their different prime factors, each taken only the greatest number of times it is found in any one of those quantities.*

104. *If the product of two quantities be divided by their greatest common divisor, the quotient will be their least common multiple.*

For, since the greatest common divisor of two quantities is composed of all the factors common to those quantities (Art. 96), these factors will enter twice into the product of the quantities. Hence, if the product be divided by the greatest common divisor, the quotient will contain only the factors *common* to the quantities, and those *peculiar to each* of them. Now these are the factors of the least common multiple. (Art. 103.)

105. To find the least common multiple of quantities.

1. Find the least common multiple of $6a^2bc$ and $4ab^3d$.

OPERATION.

$$\begin{array}{lcl}
6a^2bc & = & 3 \times 2 \times a^2 \times b \times c \\
4ab^3d & = & 2^2 \times a \times b^3 \times d \\
\hline
12a^2b^3cd & = & 3 \times 2^2 \times a^2 \times b^3 \times c \times d
\end{array}$$

Resolving the quantities into their several factors, we find that the different factors, each taken only the greatest number of times it enters into either of the quantities, are 3, 2^2, a^2, b^3, c, and d; and the product of these, or $12a^2b^3cd$, is the least common multiple. (Art. 103.) Hence the

RULE.

Resolve the quantities into their prime factors; and the product of these factors, taking each factor only the greatest number of times it enters into any one of the quantities, will be the least common multiple.

Show that the product of two quantities divided by their greatest common divisor gives their least common multiple. Explain the operation. Repeat the Rule.

Note 1. **When quantities are prime to each other, their product is their least common multiple.**

Note 2. When the greatest common divisor of two quantities is known, or the quantities are such as not to be readily factored by inspection, it may be the most convenient process of obtaining the least common multiple *to divide the product of the quantities by their greatest common divisor.* (Art. 104.)

Examples.

2. Find the least common multiple of $2a^2x$, $4ax$, and $3x - 5x^2$. Ans. $4a^2x(3 - 5x)$.

$$\begin{array}{l}
2a^2x = 2 \times a^2 \times x \\
4ax = 2^2 \times a \times x \\
3x - 5x^2 = \qquad x \times (3 - 5x) \\
\hline
4a^2x(3 - 5x) = 2^2 \times a^2 \times x \times (3 - 5x).
\end{array}$$

3. Find the least common multiple of $x^2 - a^2$ and $x^3 - a^3$. Ans. $(x + a)(x^3 - a^3)$.

$$\begin{array}{c}
x^2 - a^2 = (x - a)(x + a) \\
x^3 - a^3 = (x - a)(x^2 + ax + a^2) \\
\hline
(x + a)(x^3 - a^3) = (x + a)(x - a)(x^2 + ax + a^2)
\end{array}$$

4. Find the least common multiple of $x^2 - x - 12$ and $x^3 + 6x^2 + 9x$, their greatest common divisor being $x + 3$. Ans. $x(x - 4)(x + 3)^2$.

$$\frac{(x^2 - x - 12)(x^3 + 6x^2 + 9x)}{x + 3} = x(x - 4)(x + 3)^2$$

5. Find the least common multiple of $9x^3y^5$, $15xy^2$, and $18x^5y^6$. Ans. $90x^5y^6$.

6. Find the least common multiple of $4abc^2$, $6a^2c^3$, and $9ab^5d$.

7. Find the least common multiple of $5a^3b^2$ and $10a^3c^2(a + b)$. Ans. $10a^3b^2c^2(a + b)$.

What is Note 1? Note 2? Explain the operation of Example 2. Example 3. Example 4.

8. Find the least common multiple of $ab(x+y)$ and $ac^2(x^3+y^3)$. Ans. $abc^2(x^3+y^3)$.

9. Find the least common multiple of $3a+1$ and $3(9a^2-1)$. Ans. $3(9a^2-1)$.

10. Find the least common multiple of $1+a$, $1-a$, and $1-a^2$. Ans. $1-a^2$.

11. Required the least common multiple of a, $x+y$, and $x-y$. Ans. $a(x^2-y^2)$.

12. Required the least common multiple of

$$3a^2x+6abx+3b^2x \text{ and } 12a^2-12ab+3b^2.$$

Ans. $3x(a+b)^2(2a-b)^2$.

FRACTIONS.

106. A Fraction, in Algebra, is an algebraic expression denoting one or more equal parts of a unit.

107. A Fractional Unit is one of the equal parts into which a unit has been divided. Thus, $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{b}$ are fractional units.

108. The Denominator of a fraction shows into how many equal parts a unit has been divided, in order to produce the fractional unit.

109. The Numerator of a fraction shows how many fractional units have been taken.

110. The Terms of a fraction are its numerator and denominator.

Algebraic fractions are written like common fractions in Arithmetic, the quantity representing the numerator being placed above a horizontal line, and that representing the denominator being placed below.

Define a Fraction. A Fractional Unit. The Denominator of a fraction. The Numerator. The Terms. How are fractions written?

Thus, $\frac{a}{b}$ represents a fraction, of which a is the numerator, and b the denominator.

111. An ENTIRE ALGEBRAIC QUANTITY is one which has no fractional part; as, $a\,b$, or $a - b$.

112. A MIXED ALGEBRAIC QUANTITY is one having both entire and fractional parts; as, $a - \frac{b}{c}$, or $c + \frac{a}{x+y}$.

113. The VALUE of a fraction is the quotient arising from the division of the numerator by the denominator.

For, a fraction is an expression of division, the numerator answering to the dividend, and the denominator to the divisor. (Art. 66.) Thus, the value of the fraction $\frac{a\,b}{b}$ is a.

GENERAL PRINCIPLES OF FRACTIONS.

114. *If the numerator be multiplied, or the denominator divided, by any quantity, the fraction is multiplied by the same quantity.*

For, let $\frac{a\,b}{b}$ denote any fraction; then,

$$\frac{a\,b}{b} = a.$$

Now, if we multiply its numerator by any quantity b, we have,

$$\frac{a\,b^2}{b} = a\,b,$$

and, in like manner, if we divide its denominator by b, we obtain also $a\,b$. Hence, in both cases the value of the fraction has been multiplied by b.

115. *If the denominator be multiplied, or the numerator divided, by any quantity, the fraction is divided by the same quantity.*

Define an Entire Algebraic Quantity. A Mixed Algebraic Quantity. The Value of a fraction. If the numerator be multiplied, or the denominator divided, how is the fraction affected? If the denominator be multiplied, or the numerator divided?

For, let $\frac{ab^2}{b}$ denote any fraction; then,

$$\frac{ab^2}{b} = ab.$$

Now, if we multiply its denominator by any quantity b, we have,

$$\frac{ab^2}{b^2} = a,$$

and, in like manner, if we divide its numerator by b, we obtain also a. Hence, in both cases the value of the fraction has been divided by b.

116. *If the numerator and denominator be both multiplied, or both divided, by the same quantity, the value of the fraction will not be changed.*

For, let $\frac{ab}{b}$ denote any fraction; then,

$$\frac{ab}{b} = a.$$

Now, if we multiply both its numerator and denominator by the same quantity b, we have,

$$\frac{ab^2}{b^2} = a,$$

and, in like manner, if we divide both terms by b, we obtain also a. Hence, the value of the fraction in both cases remains unchanged.

SIGNS OF FRACTIONS.

117. *A fraction is* POSITIVE *when its numerator and denominator have the* SAME *sign, and* NEGATIVE *when they have* DIFFERENT *signs.*

For, a fraction represents the quotient of its numerator divided by its denominator, consequently its proper sign must be determined as in division. (Art. 67.)

If both numerator and denominator be multiplied or divided? When is a fraction positive? When negative?

118. The SIGN of a fraction, or that prefixed to its dividing line, shows whether the fraction is to be added or subtracted.

Thus, in $x + \frac{-a}{b}$, the sign $+$ denotes that $\frac{-a}{b}$, although essentially *negative* (Art. 117), is to be added to x.

119. The sign written before the dividing line has been termed the *apparent sign* of the fraction, and that depending upon the value expressed by the fraction itself has been termed the *real sign*. Thus, in $+\frac{-a}{b}$, the apparent sign is $+$, and the real sign $-$.

120. *If any one of the signs prefixed to the numerator, denominator, and dividing line of a fraction be changed, the value of the fraction will be changed accordingly.*

Thus, $\frac{+ab}{b} = a$; $\frac{-ab}{b} = -a$; $\frac{ab}{-b} = -a$; $-\frac{ab}{b} = -a$.

Also, $-\frac{-ab}{-b} = -a$; $-\frac{ab}{-b} = +a$; $-\frac{-ab}{b} = +a$; $\frac{-ab}{-b} = +a$.

121. *Any two of the signs prefixed to the numerator, denominator, and dividing line of a fraction may be changed, without affecting the value of the fraction.*

$$\text{Thus, } \frac{ab}{b} = -\frac{ab}{-b} = \frac{-ab}{-b} = -\frac{-ab}{b} = +a.$$

$$\text{Also, } -\frac{ab}{b} = \frac{-ab}{b} = \frac{ab}{-b} = -\frac{-ab}{-b} = -a.$$

122. *If all the signs prefixed to the terms and the dividing line of a fraction be changed, the value of the fraction will be changed accordingly.*

What does the Sign of a fraction show? What is the apparent sign of a fraction? The real sign? What is the effect of changing one of the signs prefixed to the fraction and its terms? Of changing two of the signs? Of changing all the signs?

Thus, $\frac{ab}{b} = +a$, but $-\frac{-ab}{-b} = -a$; $\frac{-ab}{b} = -a$, but $-\frac{ab}{-b} = +a$; $\frac{ab}{-b} = -a$, but $-\frac{-ab}{b} = +a$; $-\frac{ab}{b} = -a$, but $\frac{-ab}{-b} = +a$.

REDUCTION.

123. Reduction of Fractions is the process of changing their forms without altering their values.

CASE I.

124. To reduce a fraction to its lowest terms.

A fraction is in its *lowest terms*, when its terms are prime to each other.

1. Reduce $\frac{6\,ab}{9\,bc}$ to its lowest terms.

OPERATION.

$$\frac{6ab}{9bc} = \frac{3b \times 2a}{3b \times 3c} = \frac{2a}{3c}$$

We factor both terms. Then, since dividing both numerator and denominator by the same quantity does not affect the value of the fraction (Art. 116), we strike from each the common factors 3 and b, or $3b$. But $3b$ is the greatest common divisor of the terms of the fraction, consequently, $2a$ and $3c$ are prime to each other (Art. 83), and $\frac{2a}{3c}$ is the answer required.

RULE.

Resolve both terms of the fraction into their prime factors, and cancel all that are common to both. Or,

Divide both terms by their greatest common divisor.

Define Reduction of Fractions. Explain the operation. Repeat the Rule.

EXAMPLES.

2. Reduce $\frac{a^2x-x^3}{a^4-x^4}$ to its lowest

$$\frac{a^2x-x^3}{a^4-x^4}=\frac{x(a+x)(a-}{(a+x)(a-x)(a^2}$$

3. Reduce $\frac{4mny}{6m^2nx}$ to its lowest te

4. Reduce $\frac{5abx^2}{10bx}$ to its lowest terms. Ans. $\frac{ax}{2}$.

5. Reduce $\frac{3x^2y^2}{8xy^3}$ to its lowest terms.

6. Reduce $\frac{7mnx^2}{21m^2nx}$ to its lowest terms. Ans. $\frac{x}{3m}$.

7. Reduce $\frac{5abx^2}{15abm^2}$ to its lowest terms. Ans. $\frac{x^2}{3m^2}$.

8. Reduce $\frac{a^3-ab^2}{a^2+2ab+b^2}$ to its lowest terms.

Ans. $\frac{a^2-ab}{a+b}$.

9. Reduce $\frac{x^2-1}{2xy+2y}$ to its lowest terms. Ans. $\frac{x-1}{2y}$.

10. Reduce $\frac{ax+x^2}{ac^2+c^2x}$ to its lowest terms. Ans. $\frac{x}{c^2}$.

11. Reduce $\frac{x^3-a^2x}{x^2+2ax+a^2}$ to its lowest terms.

Ans. $\frac{x(x-a)}{x+a}$.

12. Reduce $\frac{a^4-a^2b^2}{a^4-b^4}$ to its lowest terms. Ans. $\frac{a^2}{a^2+b^2}$.

13. Reduce $\frac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$ to its lowest terms.

Ans. $\frac{5a^4+5a^3x}{a^2x+ax^2+x^3}$.

Thus, $a\,b$

CASE II.

95. To reduce a fraction to an entire or mixed quantity.

1. Reduce $\frac{a\,b+c}{a}$ to a mixed quantity.

OPERATION.

$$\frac{a\,b+c}{a}=b+\frac{c}{a}$$

We perform the division indicated, and obtain b for the entire part, and $+\frac{c}{a}$ for the fractional part, of the quotient. Hence the

RULE.

Divide the numerator by the denominator, for the entire part; and, if there be a remainder, write it over the denominator, for the fractional part, which connect with the entire part, by its proper sign.

EXAMPLES.

2. Reduce $\frac{a\,b+b^2}{a}$ to a mixed quantity. Ans. $b+\frac{b^2}{a}$.

3. Reduce $\frac{x^2-y^2}{x+y}$ to an entire quantity. Ans. $x-y$.

4. Reduce $\frac{a\,x+2\,x^2}{a+x}$ to a mixed quantity.

Ans. $x+\frac{x^2}{a+x}$.

5. Reduce $\frac{x^3-y^3}{x-y}$ to an entire quantity.

Ans. $x^2+x\,y+y^2$.

6. Reduce $\frac{12\,x^2-18}{3\,x}$ to a mixed quantity.

7. Change $\frac{4\,x^2-2\,x}{2\,x^2-x+1}$ to a mixed quantity.

Ans. $2-\frac{2}{2\,x^2-x+1}$.

Explain the operation. Repeat the Rule.

8. Change $\frac{a^3 - b^3 + x^3}{a + x}$ to an equivalent mixed quantity.

Ans. $a^2 - ax + x^2 - \frac{b^3}{a + x}$. e.

126. By means of negative exponents, the value of any fraction whatever may be expressed in the form of an entire quantity.

For, since any fraction is an expression of division,

Any factor may be transferred from either term of a fraction to the other by changing the sign of its exponent. (Art. 71.)

1. Reduce $\frac{4a^4b^2}{2a^3b^3c^2}$ to the form of an entire quantity.

OPERATION.

$$\frac{4a^4b^2}{2a^3b^3c^2} = \frac{2a}{bc^2} = 2ab^{-1}c^{-2}$$

We reduce the given fraction to its lowest terms, by canceling all common factors, and obtain $\frac{2a}{bc^2}$. This expression we change to the form of an entire quantity by transferring to the numerator the factors of the denominator, with the signs of their exponents changed from positive to negative, and $\frac{2a}{bc^2}$ becomes $2ab^{-1}c^{-2}$.

2. Reduce $\frac{a^2b^2}{c^3d}$ to the form of an entire quantity.

Ans. $a^2b^2c^{-3}d^{-1}$.

3. Change $\frac{5ab^2}{3b^3c^2}$ to the form of an entire quantity.

Ans. $5 \times 3^{-1}ab^{-1}c^{-2}$.

4. Change $\frac{3xy^2}{3x^2y}$ to the form of an entire quantity.

5. Change $\frac{x^2y}{a^{-1}b^2}$ to the form of an entire quantity.

Ans. $ab^{-2}x^2y$.

By what means may any fraction be expressed in the form of an entire quantity? How may any factor be transferred from one term of a fraction to the other?

6. Reduce $\frac{x^2 y^3}{a^{-3} y^{-2}}$ to the form of an entire quantity.
Ans. $a^3 x^2 y^5$.

7. Reduce $\frac{a+b}{(a-b)^2}$ to the form of an entire quantity.
Ans. $(a+b)(a-b)^{-2}$.

8. Reduce $\frac{a-b}{(a+b)^{-1}}$ to the form of an entire quantity.
Ans. $a^2 - b^2$.

9. In $\frac{4a^{-3} b^3 x^{-2}}{2a^{-2} b^{-2}}$, remove the negative exponents.
Ans. $\frac{2b^5}{ax^2}$.

10. Change $\frac{xy(a-b)^{-2}}{a+b}$ to an equivalent expression having positive exponents. Ans. $\frac{xy}{a^3 - a^2 b - ab^2 + b^3}$.

CASE III.

127. To reduce a mixed quantity to a fractional form.

1. Reduce $b + \frac{c}{a}$ to a fractional form.

OPERATION.

$$b + \frac{c}{a} = \frac{ab + c}{a}$$

Since any quantity may be expressed in the form of a fraction by writing 1 beneath it, the entire part b is the same as $\frac{b}{1}$; now, if we multiply the terms by a, the denominator of the fractional part, which will not change the value represented (Art. 116), we have $\frac{b}{1} = \frac{ab}{a}$. Then, since $\frac{ab}{a}$ and $\frac{c}{a}$ have the same fractional unit (Art. 107), we unite their numerators by the proper sign, and write the result as a numerator of a fraction of which a is the denominator, thus obtaining $\frac{ab+c}{a}$. Hence the

---

Explain the operation.

RULE.

Multiply the entire part by the denominator of the fraction; add the numerator to the product when the sign of the fraction is plus, and subtract it when the sign is minus, and write the result over the denominator.

NOTE. It is also obvious, from the analysis of the example preceding the rule, that any entire quantity may be reduced to a fractional form having a GIVEN DENOMINATOR, by *multiplying the entire quantity by the given denominator, and writing the product over that denominator.*

EXAMPLES.

2. Reduce $x - \frac{a^2 - x^2}{x}$ to a fractional form.

$$x - \frac{a^2 - x^2}{x} = \frac{x^2 - (a^2 - x^2)}{x} = \frac{x^2 - a^2 + x^2}{x} = \frac{2x^2 - a^2}{x}.$$

3. Reduce $x - \frac{a^2 b^2}{x}$ to a fractional form.

Ans. $\frac{x^2 - a^2 b^2}{x}$.

4. Reduce $b + \frac{a^2 x^2}{a}$ to a fractional form.

5. Reduce $a - \frac{ab - a^2}{2b}$ to a fractional form.

Ans. $\frac{ab + a^2}{2b}$.

6. Reduce $a + 1 - \frac{x - 1}{b}$ to a fractional form.

Ans. $\frac{ab + b - x + 1}{b}$.

7. Reduce $2a - 2b + \frac{a - x}{3}$ to a fractional form.

Ans. $\frac{7a - 6b - x}{3}$.

Repeat the Rule. How may an entire quantity be reduced to a fractional form having a given denominator?

8. Change $1 + 3a - \frac{4x - 5}{4x}$ to an equivalent fractional form. Ans. $\frac{12ax + 5}{4x}$.

9. Change $a + b - \frac{a^2 - b^2 - 3}{a - b}$ to an equivalent fractional form. Ans. $\frac{3}{a - b}$.

10. Change $2 + \frac{x^2 + y^2}{xy}$ to an equivalent fractional form.

Ans. $\frac{(x + y)^2}{xy}$.

11. Reduce $a + b - \frac{a^2 - 2ab + b^2}{a + b}$ to the form of a fraction. Ans. $\frac{4ab}{a + b}$.

CASE IV.

128. To reduce fractions of different denominators to equivalent fractions having a common denominator.

Fractions are said to have a common denominator when they have the same quantity for a denominator.

1. Reduce $\frac{a}{b}$ and $\frac{c}{d}$ to a common denominator.

OPERATION.

$$\frac{a}{b} = \frac{a \times d}{b \times d} = \frac{ad}{bd}$$

$$\frac{c}{d} = \frac{c \times b}{d \times b} = \frac{bc}{bd}$$

We multiply both terms of each fraction by the denominator of the other, which does not change the value of the fraction, since both terms have been multiplied by the same quantity (Art. 116), and have as equivalent to $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{ad}{bd}$ and $\frac{bc}{bd}$, respectively, with a common denominator, bd.

Now, the common denominator, bd, since it is divisible by each of the given denominators, b and d, is a *common multiple* of them

---

Explain the first operation.

(Art. 102); and it will also be noticed that each numerator was multiplied by a quantity equal to the quotient resulting from dividing this multiple by its denominator.

2. Reduce $\frac{a}{xy}$, $\frac{ax}{y}$, and $\frac{a}{x}$ to equivalent fractions having the least common denominator.

OPERATION.

$$(xy \div xy) \times a = a; \quad \frac{a}{xy} = \frac{a}{xy}$$

$$(xy \div y) \times ax = ax^2; \quad \frac{ax}{y} = \frac{ax^2}{xy}$$

$$(xy \div x) \times a = ay; \quad \frac{a}{x} = \frac{ay}{xy}$$

Since, as has been shown, a common multiple of the denominators of the given fractions will be a common denominator of the required fractions, the least common multiple of the denominators will be the least common denominator. The least common multiple of the denominators we find to be xy; it is, consequently, the least common denominator of the required fractions.

We next divide the least common denominator by each of the given denominators, and ascertain that the multipliers required to change each to the least common denominator are 1, x, and y. As the denominators are to be multiplied by these quantities, respectively, the numerators must be multiplied by the same, that the value of the fractions may not be changed (Art. 116), and we thus obtain the new numerators, a, ax^2, and ay. These, written over the least common denominator, xy, give $\frac{a}{xy}$, $\frac{ax^2}{xy}$ and $\frac{ay}{xy}$, the fractions required.

RULE.

Multiply each numerator by all the denominators except its own, for new numerators, and all the denominators together for a COMMON *denominator.* Or,

Find the least common multiple of all the denominators for the LEAST COMMON *denominator. Divide this multiple by each denominator, separately, and multiply the quotients by the corresponding numerators for new numerators.*

Explain the second operation. Repeat the Rule.

Note 1. Entire quantities should be reduced to a fractional form by writing 1 for a denominator to each, when required to be reduced to a common denominator with fractions.

Note 2. All the denominators, if necessary, should be made positive (Art. 121) before finding a common denominator, and the fractions should be reduced to their lowest terms before finding the least common denominator.

Examples.

3. Reduce $\frac{2x}{a}$ and $\frac{c}{n}$ to equivalent fractions having a common denominator. Ans. $\frac{2nx}{an}, \frac{ac}{an}$.

4. Reduce $\frac{4x}{5y}$, $\frac{7m}{10y^2}$, and $\frac{n}{2x}$, to equivalent fractions having the least common denominator.
Ans. $\frac{8x^2y}{10xy^2}, \frac{7mx}{10xy^2}, \frac{5ny^2}{10xy^2}$.

5. Reduce $\frac{3x}{2a}$, $\frac{2x}{5a^2}$, and $\frac{m}{n}$ to equivalent fractions having the least common denominator.

6. Reduce $\frac{3}{4}$, $\frac{2x}{3}$, and $a + \frac{4x}{5}$ to equivalent fractions having the least common denominator.
Ans. $\frac{45}{60}, \frac{40x}{60}, \frac{60a + 48x}{60}$.

7. Reduce $\frac{2x+3}{x}$ and $\frac{5x+1}{3}$ to equivalent fractions having a common denominator.
Ans. $\frac{6x+9}{3x}, \frac{5x^2+x}{3x}$.

8. Reduce a, $\frac{a}{b}$, and $\frac{x-2}{c+d}$ to equivalent fractions having a common denominator.
Ans. $\frac{abc+abd}{bc+bd}, \frac{ac+ad}{bc+bd}, \frac{bx-2b}{bc+bd}$.

What is Note 1? Note 2?

129. Fractions may often be very readily reduced to equivalent ones, having a common denominator, by multiplying both numerator and denominator of one or more of them, so as to make the denominator the same for each, the multiplier being determined by inspection.

1. Reduce $\frac{2x}{x^2-a^2}$ and $\frac{3}{x+a}$ to equivalent fractions having the least common denominator.

$$\frac{2x}{x^2-a^2},\ \frac{3}{x+a}=\frac{2x}{x^2-a^2},\ \frac{3(x-a)}{x^2-a^2}$$

Since we know that $(x+a)(x-a)=x^2-a^2$, we convert the second fraction into one with a denominator the same as the first, by multiplying both terms of the second by $x-a$.

2. Change $\frac{a}{x+y}$, $\frac{b}{x-y}$, and $\frac{c}{x^2-y^2}$ into equivalent fractions having the least common denominator.

Ans. $\frac{a(x-y)}{x^2-y^2},\ \frac{b(x+y)}{x^2-y^2},\ \frac{c}{x^2-y^2}$.

3. Change $\frac{a+x}{a^3-x^3}$, $\frac{c}{a-x}$, and $\frac{bc}{a^2+ax+x^2}$ into equivalent fractions having the least common denominator.

Ans. $\frac{a+x}{a^3-x^3},\ \frac{c(a^2+ax+x^2)}{a^3-x^3},\ \frac{bc(a-x)}{a^3-x^3}$.

ADDITION.

130. Addition of Fractions is the process of collecting two or more fractional quantities into one equivalent expression, called the sum.

Since only quantities of the same unit value can be united in one sum, fractions to be added must express fractional units of the same kind, or have a common denominator.

How may fractions often be reduced to those having a common denominator? Define Addition of Fractions. Why must fractions to be added have a common denominator?

131. To add fractions.

1. What is the sum of $\frac{a}{b}$, $\frac{c}{b}$, and $\frac{d}{b}$?

OPERATION.

$$\frac{a}{b}+\frac{c}{b}+\frac{d}{b}=\frac{a+c+d}{b}$$

Since the fractions, by having a common denominator, express fractional units of the same kind, we add them by writing the sum of their numerators, $a+c+d$, over the common denominator, b, and obtain $\frac{a+c+d}{b}$.

2. What is the sum of $\frac{a}{b}$ and $\frac{c}{d}$?

OPERATION.

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$$

We reduce the given fractions to equivalent ones having a common denominator, and then find the sum as in the preceding example.

RULE.

Reduce the fractions, if necessary, to equivalent ones having a common denominator. Add the numerators, and write the sum over the common denominator.

NOTE. The final result should be reduced to its lowest terms, whenever any reductions are possible.

EXAMPLES.

3. What is the sum of $\frac{3a}{4}$, $\frac{5a}{6}$, and $\frac{a}{3}$? Ans. $\frac{23a}{12}$.

4. What is the sum of $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$? Ans. $\frac{13x}{12}$.

5. What is the sum of $\frac{3x}{2a}$ and $\frac{x}{5}$?

Ans. $\frac{15x+2ax}{10a}$.

Explain the first operation. The second. Repeat the Rule.

6. Add $\frac{4adm}{8bdm}$, $\frac{2bcm}{8bdm}$, and $\frac{3bdc}{8bdm}$.

Ans. $\frac{4adm + 2bcm + 3bdc}{8bdm}$.

7. Add $\frac{4x}{7}$ and $\frac{x-2}{5}$. Ans. $\frac{27x-14}{35}$.

8. Required the sum of $\frac{3x-a}{14}$ and $\frac{4x}{7}$.

9. Add $\frac{x}{x+y}$ and $\frac{y}{x-y}$. Ans. $\frac{x^2+y^2}{x^2-y^2}$.

10. Add $\frac{1}{1+a}$ and $\frac{1}{1-a}$. Ans. $\frac{2}{1-a^2}$.

11. Add $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$. Ans. $\frac{2(x^2+y^2)}{x^2-y^2}$.

12. Required the sum of $\frac{a-b}{ab}$, $\frac{b-c}{bc}$, and $\frac{c-a}{ac}$.

Ans. 0.

13. Add $\frac{x-a}{x^2-ax+a^2}$ and $\frac{1}{x+a}$. Ans. $\frac{2x^2-ax}{x^3+a^3}$.

14. Add $\frac{n}{n-1}$ and $\frac{1-2n}{n^2-n}$. Ans. $\frac{n-1}{n}$.

Note. When entire or mixed quantities and fractions are to be added, the entire quantities and the fractions may be added separately; or the mixed quantities may be reduced to the form of a fraction before adding.

15. Add $5a$, $4a + \frac{c}{b}$, and $\frac{d}{b}$. Ans. $9a + \frac{c+d}{b}$.

16. Add $3a + \frac{2x}{5}$ and $a - \frac{8x}{9}$. Ans. $4a - \frac{22x}{45}$.

17. What is the sum of c, $\frac{c+d}{2}$, and $\frac{c-d}{2}$?

Ans. $2c$.

18. What is the sum of $x - \frac{4a^2}{b}$ and $y + \frac{2ax}{c}$?

Ans. $x + y + \frac{2abx - 4a^2c}{bc}$.

How may entire or mixed quantities and fractions be added?

SUBTRACTION.

132. Subtraction of Fractions is the process of finding the difference between two fractions.

Since one quantity can be subtracted from another only when they have the same unit value, one fraction can be subtracted from another only when they express fractional units of the same kind, or have a common denominator.

133. To subtract one fraction from another.

1. Subtract $\frac{c}{b}$ from $\frac{a}{b}$.

OPERATION.

$$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

Since the fractions, by having a common denominator, express fractional units of the same kind, we subtract the fraction $\frac{c}{b}$ from $\frac{a}{b}$ by writing the difference of their numerators, $a-c$, over the common denominator, b, and obtain $\frac{a-c}{b}$.

2. Subtract $\frac{ax}{b+c}$ from $\frac{ax}{b-c}$.

OPERATION.

$$\frac{ax}{b-c}-\frac{ax}{b+c}=\frac{abx+acx}{b^2-c^2}-\frac{abx-acx}{b^2-c^2}$$

$$=\frac{abx+acx-(abx-acx)}{b^2-c^2}=\frac{2acx}{b^2-c^2}$$

We reduce the fractions to equivalent ones having a common denominator; then, subtracting the numerator of the subtrahend from that of the minuend, writing the difference over the common denominator, and reducing, we obtain $\frac{2acx}{b^2-c^2}$. Hence the

---

Define Subtraction of Fractions. When can one fraction be subtracted from another? Explain the first operation. The second.

RULE.

Reduce the fractions, if necessary *uantity.*
common denominator. Subtract the
from that of the minuend, and writ
mon denominator.

Note. When there are entire or mixe
fractions, the former may either be reduc
tracted separately.

Examples.

3. From $\frac{12x}{7}$ take $\frac{3x}{5}$. Ans. $\frac{39x}{35}$.

4. From $\frac{a}{3}$ take $\frac{a}{4}$. Ans. $\frac{a}{12}$.

5. From $\frac{3ab}{4}$ take $\frac{4ab}{2}$. Ans. $-\frac{5ab}{4}$.

6. From $\frac{x+y}{2}$ take $\frac{x-y}{2}$. Ans. y.

7. Subtract $\frac{1}{x+1}$ from $\frac{1}{x-1}$.

8. Subtract $\frac{n}{n-1}$ from $\frac{1-2n}{n^2-n}$. Ans. $\frac{1-2n-n^2}{n^2-n}$.

9. From $3x-\frac{x}{2b}$ take $x-\frac{x-a}{c}$.
Ans. $\frac{4bcx+2bx-cx-2ab}{2bc}$.

10. From $\frac{1+x^2}{1-x^2}$ take $\frac{4x^2}{1-x^4}$. Ans. $\frac{1-x^2}{1+x^2}$.

11. From $4x+\frac{b}{a}$ take $3x-\frac{c}{d}$. Ans. $x+\frac{bd+ac}{ad}$.

12. From $7b-\frac{4a+1}{2}$ take $b+\frac{3}{5}$.
Ans. $6b-\frac{20a+11}{10}$.

Repeat the Rule. The Note.

m $\frac{3x+2}{y}$ take $\frac{7xy-10y}{y^2}$. Ans. $\frac{4(3-x)}{y}$.

132.

ing What is the value of $1-\frac{x-a}{x+a}$? Ans. $\frac{2a}{x+a}$.

15. What is the value of $2x+\frac{3x}{a}-x-\frac{2x-2a}{3c}$?

Ans. $x+\frac{9cx-2ax+2a^2}{3ac}$.

16. From $\frac{xy}{x-y}$ subtract $\frac{xy}{x+y}$. Ans. $\frac{2xy^2}{x^2-y^2}$.

MULTIPLICATION.

134. Multiplication of Fractions is the process of multiplying when one or both of the factors are fractions.

CASE I.

135. To multiply a fraction by an entire quantity.

1. Multiply $\frac{a}{b}$ by c.

OPERATION.

$$\frac{a}{b}\times c=\frac{ac}{b}$$

Since a fraction is multiplied by multiplying its numerator (Art. 114), we multiply the numerator, a, by c, and obtain $\frac{ac}{b}$.

2. Multiply $\frac{a}{b^2}$ by b.

OPERATION.

$$\frac{a}{b^2}\times b=\frac{a}{b}$$

Since a fraction is multiplied by dividing its denominator (Art. 114), we divide the denominator, b^2, by b, and obtain $\frac{a}{b}$.

Define Multiplication of Fractions. Explain the first operation. The second.

RULE.

Multiply the numerator of the fraction by the entire quantity. Or,

Divide the denominator of the fraction by the entire quantity.

Note. The second method is preferable when the entire quantity will divide the denominator without a remainder.

Examples.

3. Multiply $\frac{x}{y}$ by a. Ans. $\frac{a\,x}{y}$.

4. Multiply $\frac{3\,a}{c\,d}$ by $a\,b$. Ans. $\frac{3\,a^2\,b}{c\,d}$.

5. Multiply $\frac{a\,y^2}{6\,x}$ by $3\,x$. Ans. $\frac{a\,y^2}{2}$.

6. Multiply $\frac{a+b}{c}$ by $a\,d$. Ans. $\frac{a^2\,d+a\,b\,d}{c}$.

7. Multiply $\frac{a+c}{1+x}$ by $a-c$. Ans. $\frac{a^2-c^2}{1+x}$.

8. Multiply $\frac{b-c}{a\,b+a\,c+b\,c+c^2}$ by $a+c$.
Ans. $\frac{b-c}{b+c}$.

9. Multiply $\frac{a+b+c}{9\,(x-y)\,(x+y)}$ by $3\,(x+y)$.
Ans. $\frac{a+b+c}{3\,(x-y)}$.

136. It is evident, from the second rule of the preceding article, that multiplying a fraction by a quantity equal to its denominator cancels the denominator, and gives the numerator for the product. Hence,

If a fraction be multiplied by any multiple of its denominator, the product will be an entire quantity.

Repeat the Rule. The Note. What is the effect of multiplying a fraction by a quantity equal to its denominator, or a multiple of it?

1. Multiply $\frac{a}{b}$ by b. Ans. a.

2. Multiply $\frac{a x}{2 b y}$ by $8 b y$. Ans. $4 a x$.

3. Multiply $\frac{x y^2}{a - b}$ by $a^2 - b^2$. Ans. $x y^2 (a + b)$.

CASE II.

137. **To multiply an entire quantity, or a fraction, by a fraction.**

1. Multiply c by $\frac{a}{b}$.

FIRST OPERATION.

$$c \times \frac{a}{b} = \frac{a}{b} \times c = \frac{a c}{b}$$

Since the product of two quantities is the same, whichever be taken for the multiplier (Art. 58), $c \times \frac{a}{b}$ is the same as $\frac{a}{b} \times c$; and by Case I. $\frac{a}{b} \times c$ is equal to $\frac{a c}{b}$.

SECOND OPERATION.

$$c \times \frac{a}{b} = c \times a b^{-1} = a c b^{-1} = \frac{a c}{b}$$

Since $\frac{a}{b}$ is equal to $a b^{-1}$ (Art. 126), $c \times \frac{a}{b}$ is equal to $c \times a b^{-1}$, or $a c b^{-1}$, which, by transferring the factor b^{-1} to the denominator (Art. 126), gives, as before, $\frac{a c}{b}$.

2. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

FIRST OPERATION.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a c}{b d}$$

We first multiply $\frac{a}{b}$ by c, and obtain $\frac{a c}{b}$; but this result is too great, since the proposed multiplier was not

---

Explain the operations of the first Example. The second Example.

c, but c divided by d, or $\frac{c}{d}$; consequently $\frac{ac}{b}$ must be divided by d, which we do by multiplying its denominator (Art. 115), and obtain $\frac{ac}{bd}$.

SECOND OPERATION.

$$\frac{a}{b} \times \frac{c}{d} = ab^{-1} \times cd^{-1} = acb^{-1}d^{-1} = \frac{ac}{bd}$$

Since $\frac{a}{b}$ is equal to ab^{-1} and $\frac{c}{d}$ to cd^{-1} (Art. 126), $\frac{a}{b} \times \frac{c}{d}$ is equal to $ab^{-1} \times cd^{-1}$, or $acb^{-1}d^{-1}$; which, by transferring the factors b^{-1} and d^{-1} to the denominator (Art. 126), gives, as before, $\frac{ac}{bd}$.

In the first example, it is evident, since c is the same as $\frac{c}{1}$, that the operation might have been performed in the same manner as in the second example. Hence the

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator.

NOTE 1. When either of the factors is an entire or mixed quantity, it may be best to reduce it to an equivalent fractional form.

NOTE 2. When there are common factors in the numerators and denominators, they may be canceled before performing the multiplication, as the result should always be expressed in its lowest terms.

EXAMPLES.

3. Multiply $\frac{x}{y}$ by $\frac{a}{m}$. Ans. $\frac{ax}{my}$.

4. Multiply $\frac{4am}{5}$ by $\frac{b}{c}$. Ans. $\frac{4abm}{5c}$.

5. Multiply $\frac{5m}{3}$ by $\frac{m}{n}$. Ans. $\frac{5m^2}{3n}$.

Repeat the Rule. What is Note 1? Note 2?

6. Required the product of $\frac{bm^2}{a}$ by $\frac{c}{d}$.

7. Multiply $4xy$ by $\frac{5ab}{4xy}$. Ans. $5ab$.

8. Multiply $\frac{2ax}{by}$ by $\frac{by}{2ax}$. Ans. 1.

9. Multiply $\frac{a-b}{3}$ by $\frac{6c}{x(a-b)}$. Ans. $\frac{2c}{x}$.

10. Multiply $\frac{(a-b)^2}{a+b}$ by $\frac{b}{a-b}$. Ans. $\frac{b(a-b)}{a+b}$.

11. Multiply $a+\frac{x}{y}$ by $\frac{y}{a}$. Ans. $\frac{ay+x}{a}$.

12. Multiply $\frac{a+b}{ax}$ by $\frac{4ax^2}{6}$. Ans. $\frac{2x(a+b)}{3}$.

13. Multiply $\frac{4x+2}{3}$ by $\frac{5x}{2x+1}$. Ans. $\frac{10x}{3}$.

14. Multiply $\frac{a+b}{b}$ by $\frac{b^2x}{a^2-b^2}$. Ans. $\frac{bx}{a-b}$.

15. Multiply $\frac{(x+y)^3}{3a}$ by $\frac{3a^2}{x+y}$. Ans. $a(x+y)^2$.

16. Multiply $\frac{a^2+b^2}{a^2-b^2}$ by $\frac{a-b}{a+b}$. Ans. $\frac{a^2+b^2}{a^2+2ab+b^2}$.

17. Required the continued product of $\frac{a-b}{a}$, $\frac{a+b}{b}$, and $\frac{a^2}{a^2-b^2}$. Ans. $\frac{a}{b}$.

18. What is the value of $\left(a-\frac{b^2}{a}\right)\left(\frac{a}{b}+\frac{b}{a}\right)$?

Ans. $\frac{a^4-b^4}{a^2b}$.

19. What is the value of $\left(\frac{x}{x+y}\right)\left(\frac{1}{x-y}\right)\left(\frac{x^2-y^2}{x}\right)$?

Ans. 1.

20. Find the product of $a+\frac{b}{y}$ by $x+\frac{m}{n}$.

Ans. $\frac{anxy+amy+bnx+bm}{ny}$.

DIVISION.

138. Division of Fractions is the process of dividing when the divisor or dividend, or both, are fractions.

CASE I.

139. To divide a fraction by an entire quantity.

1. Divide $\frac{ac}{b}$ by c.

OPERATION.

$$\frac{ac}{b} \div c = \frac{a}{b}$$

Since a fraction is divided by dividing its numerator (Art. 115), we divide the numerator, ac, by c, and obtain $\frac{a}{b}$.

2. Divide $\frac{x}{y}$ by a.

OPERATION.

$$\frac{x}{y} \div a = \frac{x}{ay}$$

Since a fraction is divided by multiplying its denominator (Art. 115), we multiply the denominator, y, by a, and obtain $\frac{x}{ay}$.

RULE.

Divide the numerator of the fraction by the entire quantity. Or,

Multiply the denominator of the fraction by the entire quantity.

Note. The first method is preferable when the entire quantity will divide the numerator without a remainder.

Examples.

3. Divide $\frac{a}{m}$ by x. Ans. $\frac{a}{mx}$.

Define Division of Fractions. Explain the first operation. The second. Repeat the Rule. What is the Note?

4. Divide $\frac{4x^2}{7}$ by $5x$. Ans. $\frac{4x}{35}$.

5. Divide $\frac{18x^2y}{3m}$ by $6x^2$. Ans. $\frac{y}{m}$.

6. Divide $\frac{x+xy}{1+y}$ by x. Ans. 1.

7. Divide $\frac{abc+b^2c}{x}$ by $a+b$. Ans. $\frac{bc}{x}$.

8. Divide $\frac{a^2d+abd}{c}$ by ad.

9. Divide $\frac{b-c}{b+c}$ by $a+c$. Ans. $\frac{b-c}{ab+ac+bc+c^2}$.

CASE II.

140. To divide an entire quantity, or a fraction, by a fraction.

1. Divide a by $\frac{c}{b}$.

FIRST OPERATION.

$$a \div \frac{c}{b} = \frac{a \times b}{c} = \frac{ab}{c}$$

We first divide a by c, and obtain $\frac{a}{c}$; but this result is too small, since the proposed divisor is not c, but c divided by b, or $\frac{c}{b}$; consequently $\frac{a}{c}$ should be taken b times, which we do by multiplying its numerator by b (Art. 114), and obtain $\frac{ab}{c}$.

SECOND OPERATION.

$$a \div \frac{c}{b} = a \div cb^{-1} = \frac{a}{cb^{-1}} = \frac{ab}{c}$$

Since $\frac{c}{b}$ is equal to cb^{-1} (Art. 126), $a \div \frac{c}{b}$ is equal to $a \div cb^{-1}$, or $\frac{a}{cb^{-1}}$, which, by transferring the factor b^{-1} to the numerator (Art. 126), gives, as before, $\frac{ab}{c}$.

Explain the operations of the first Example.

2. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

FIRST OPERATION.

$$\frac{a}{b} \div \frac{c}{d} = a b^{-1} \div c d^{-1} = \frac{a b^{-1}}{c d^{-1}} = \frac{a d}{b c}$$

Since $\frac{a}{b}$ is equal to $a b^{-1}$ and $\frac{c}{d}$ to $c d^{-1}$ (Art. 126), $\frac{a}{b} \div \frac{c}{d}$ is equal to $a b^{-1} \div c d^{-1}$, or $\frac{a b^{-1}}{c d^{-1}}$; which, by transferring factors (Art. 126), gives $\frac{a d}{b c}$.

SECOND OPERATION.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a d}{b c}$$

Reducing the fractions to equivalent ones having a common denominator, we have $\frac{a d}{b d}$ to be divided by $\frac{b c}{b d}$, which gives, as before, $\frac{a d}{b c}$. Now, $\frac{a d}{b c} = \frac{a}{b} \times \frac{d}{c}$, or $\frac{a}{b}$ is multiplied by the divisor inverted. Hence the

RULE.

Invert the terms of the divisor, and proceed as in multiplication.

NOTE 1. When either of the quantities is entire or mixed, it should be reduced to a fractional form before applying the rule.

NOTE 2. After the operation is indicated, the work should be abridged, as far as possible, by canceling factors common to the numerators and denominators, so as to express the result in its lowest terms.

EXAMPLES.

3. Divide $\frac{3 a y}{4 m}$ by $\frac{2 m}{y}$. Ans. $\frac{3 a y^2}{8 m^2}$.

4. Divide $\frac{a}{2}$ by $\frac{3 m}{4 a}$. Ans. $\frac{2 a^2}{3 m}$.

Explain the operations of the second Example. Repeat the Rule. What is Note 1? Note 2?

5. Divide $\frac{4a^2}{9m}$ by $\frac{2a}{3}$. Ans. $\frac{2a}{3m}$.

6. Required the quotient of $\frac{8a}{9m}$ divided by $\frac{a}{n}$.

7. Find the quotient of ab divided by $\frac{3d}{2ab}$.

Ans. $\frac{2a^2b^2}{3d}$.

8. Divide ay by $\frac{a-b}{x}$. Ans. $\frac{axy}{a-b}$.

9. Divide $\frac{a}{1-a}$ by $\frac{a}{4}$. Ans. $\frac{4}{1-a}$.

10. Divide $\frac{a+b}{2}$ by $\frac{a-b}{2}$. Ans. $\frac{a+b}{a-b}$.

11. What is the quotient of $x+\frac{a}{y}$ by $\frac{y}{a}$?

Ans. $\frac{axy+a^2}{y^2}$.

12. What is the quotient of $\frac{3a+b}{6}$ by $\frac{ab+m}{7a}$?

Ans. $\frac{21a^2+7ab}{6ab+6m}$.

13. Divide $\frac{a^2m+bmy}{an}$ by $a+\frac{by}{a}$. Ans. $\frac{m}{n}$.

14. Divide $\frac{a+1}{a}$ by $\frac{a^2-1}{a}$. Ans. $\frac{1}{a-1}$.

15. Divide $\frac{6x^2-2x}{4-x^2}$ by $\frac{x^2}{2+x}$. Ans. $\frac{6x-2}{2x-x^2}$.

16. Divide $\frac{2b^2}{a^3+b^3}$ by $\frac{b}{a+b}$. Ans. $\frac{2b}{a^2-ab+b^2}$.

17. What is the value of $\frac{x^2-y^2}{x+2y} \div \frac{x-y}{3x+6y}$?

Ans. $3(x+y)$.

18. What is the value of $(1+x) \div \frac{1}{x}(1+x)$?

Ans. x.

19. What is the value of 12 divided by $\frac{(a+x)^2}{x} - a$?

Ans. $\frac{12x}{a^2+ax+x^2}$.

CASE III.

141. To reduce complex fractions to simple ones.

A Complex Fraction is one having a fraction in its numerator, or denominator, or in both.

142. A complex fraction may be regarded as a case in division of fractions.

1. Reduce $\dfrac{\frac{a}{c}}{\frac{d}{b}}$ to a simple fraction.

FIRST OPERATION.

$$\frac{\frac{a}{c}}{\frac{d}{b}} = \frac{a}{c} \div \frac{d}{b} = \frac{ab}{cd}$$

Since $\dfrac{\frac{a}{c}}{\frac{d}{b}}$ is the same as $\frac{a}{c} \div \frac{d}{b}$, we regard it as a case in division; and, reducing the expression by the rule in Case II., obtain the simple fraction $\frac{ab}{cd}$.

SECOND OPERATION.

$$\frac{\frac{a}{c} \times bc}{\frac{d}{b} \times bc} = \frac{ab}{cd}$$

Since multiplying a fraction by any multiple of its denominator cancels that denominator (Art. 136), we multiply both terms of the complex fraction by the least common multiple of their denominators, and obtain the simple fraction $\frac{ab}{cd}$.

Hence, to reduce a complex fraction to a simple one, or to simplify it,

---

Define a Complex Fraction. Explain the first operation. The second.

RULE.

Consider the denominator as a divisor, and the numerator as a dividend, and proceed as in Case II. Or,

Multiply both terms of the complex fraction by the least common multiple of their denominators.

NOTE. When the terms of a fraction contain negative exponents, the fraction may be regarded as a complex one. If the letter or quantity which bears the negative exponent is a *factor* of either numerator or denominator, as a whole, the negative exponent may be removed as in Art. 126; otherwise, both numerator and denominator must be multiplied by that letter or quantity with an equal positive exponent, in accordance with the above rule.

2. Reduce $\dfrac{\frac{x}{y}}{\frac{m}{n}}$ to a simple fraction. Ans. $\dfrac{nx}{my}$.

3. Reduce $\dfrac{\frac{a+b}{x}}{\frac{y-a}{b}}$ to a simple fraction.

Ans. $\dfrac{ab+b^2}{xy-ax}$.

4. Reduce $\dfrac{x}{y+\frac{a}{b}}$ to a simple fraction.

Ans. $\dfrac{bx}{by+a}$.

5. Reduce $\dfrac{\frac{5-c}{x}}{\frac{7-y}{a}}$ to a simple fraction.

Ans. $\dfrac{5a-ac}{7x-xy}$.

6. Reduce $\dfrac{y-mn^{-1}}{x+ad^{-1}}$ to a simple fraction.

Ans. $\dfrac{dny-dm}{dnx+dn}$.

How may we reduce a complex fraction to a simple one, or simplify it? What is said of fractions containing negative exponents?

7. Reduce $\frac{b}{x + a^{-1}}$ to a simple fraction.

Ans. $\frac{a\,b}{a\,x + 1}$.

8. Simplify the expression $\frac{\frac{1}{b - x}}{b + x}$. Ans. $\frac{1}{b^2 - x^2}$.

9. Simplify the expression $\frac{\frac{2\,x^2 - b}{5}}{\frac{a - b}{4}}$.

Ans. $\frac{8\,x^2 - 4\,b}{5\,a - 5\,b}$.

10. Simplify the expression $\frac{x^4 - \frac{1}{x^4}}{x + \frac{1}{x}}$.

Ans. $x^3 - x + \frac{1}{x} - \frac{1}{x^3}$.

Note. The last example furnishes a good opportunity for the use of negative exponents. Dividing $x^4 - x^{-4}$ by $x + x^{-1}$ gives $x^3 - x + x^{-1} - x^{-3}$ as a quotient. The answer given above may also be obtained by dividing $x^4 - \frac{1}{x^4}$ by $x + \frac{1}{x}$, or by simplifying the fraction according to the rule, and then reducing the fraction to a mixed number.

SIMPLE EQUATIONS.

143. An Equation is an expression of equality between two quantities. Thus,

$$x + 4 = 16$$

is an equation, expressing the equality of the quantities $x + 4$ and 16.

144. The quantity on the left of the sign of equality is called the *first member*, or *side*, and that on the right, the *second member*, or *side*, of the equation.

Define an Equation. Members or sides of an equation.

145. The DEGREE of an equation containing but one unknown quantity is denoted by the exponent of the highest power of that unknown quantity to be found in the equation. (Art. 15.) Thus,

An equation of the *first degree* is one that contains no higher power of the unknown quantity than its *first* power; as,

$$x + 14 = 28 - 4, \text{ or } cx = a^2 + bd.$$

An equation of the *second degree* is one in which the highest power of the unknown quantity is the *second* power, or square; as,

$$3x^2 - 2x = 65.$$

In like manner, we have equations of the *third degree*, *fourth degree*, and so on.

NOTE. The degree of an equation must be distinguished from the degree of its terms (Art. 25). The former depends altogether upon the unknown quantity, without any reference to the latter. Thus, the equation $cx = a^2 + bd$ is of the *first* degree, while each of its terms is of the *second*.

146. A SIMPLE EQUATION is an equation of the first degree.

147. A NUMERICAL EQUATION is one in which all the known quantities are expressed by figures; as,

$$2x - x = 17 - 5.$$

NOTE. The degree of a numerical equation corresponds with the highest degree of any of its terms.

148. A LITERAL EQUATION is one in which some or all the known quantities are expressed by letters; as,

$$2x + a = x^2 - 10.$$

149. An IDENTICAL EQUATION is one in which the two

Define the Degree of an equation. Equation of the first degree. Second degree. A Simple Equation. A Numerical Equation. A Literal Equation. An Identical Equation.

members are the same, or become the same on performing the operations indicated; as,

$$x - y = x - y, \quad \text{or} \quad 2a + 2bc = 2(a + bc).$$

TRANSFORMATION OF EQUATIONS.

150. The Transformation of an equation is the process of changing its form without destroying the equality.

151. The transformation of an equation depends upon the axioms (Art. 38), and we may, without destroying the equality, —

1. Add equal quantities to both members (Ax. 1).
2. Subtract equal quantities from both members (Ax. 2).
3. Multiply both members by the same quantity (Ax. 3).
4. Divide both members by the same quantity (Ax. 4).
5. Raise both members to the same power (Ax. 8).
6. Take the same root of both members (Ax. 8).

In the transformation of simple equations, there are two principal cases: —

I. Transposition of terms.
II. Clearing of fractions.

CASE I.

152. To transpose terms of an equation.

Transposition is the process of changing terms from one member of an equation to the other, without destroying the equality.

Define the Transformation of an equation. Upon what does the transformation depend? What are the two principal cases? Define Transposition.

1. Let it be required, in $x - a = b$, to transpose $-a$ to the second member.

OPERATION.

$$\begin{array}{r} x - a = b \\ a = a \\ \hline x = b + a \end{array}$$

Since we may add an equal quantity to both members of an equation, without destroying the equality (Art. 151), we add a to each member, and obtain $x = b + a$.

2. Let it be required, in $x + a = b$, to transpose a to the second member.

OPERATION.

$$\begin{array}{r} x + a = b \\ a = a \\ \hline x = b - a \end{array}$$

Since we may subtract an equal quantity from both members of an equation, without destroying the equality (Art. 151), we subtract a from each member, and obtain $x = b - a$.

Now, the result is the same, in each of the above operations, as if we had transferred a from the first to the second member, and *changed its sign*. Hence the

RULE.

Any term may be transposed from one member of an equation to the other, provided its sign be changed.

NOTE. It also follows, that *the signs of all the terms of an equation may be changed, without destroying the equality.*

Transpose the unknown terms to the first member, and the known terms to the second, in the following

EXAMPLES.

3. $2x - a = b$. Ans. $2x = a + b$.

4. $11x + 9 = 6x + 34$. Ans. $11x - 6x = 34 - 9$.

5. $5x + 3 = 2x + 24$.

Explain the first operation. The second. Repeat the Rule. The Note.

6. $3b + 2x - 25 = ax$. Ans. $2x - ax = 25 - 3b$.

7. $3ac - cd + xy = 6ad - 7x$.

Ans. $7x + xy = 6ad - 3ac + cd$.

CASE II.

153. To clear an equation of fractions.

1. Clear the equation $\frac{x+6}{2} - 26 = \frac{5x}{4} + 2$ of fractions.

OPERATION.

$$\frac{x+6}{2} - 26 = \frac{5x}{4} + 2$$

$$2x + 12 - 104 = 5x + 8$$

Since multiplying a fraction by any multiple of its denominator will give for the product an entire quantity (Art. 136), we multiply each term of the equation by the least common multiple of the denominators, or 4 (Art. 151), and, canceling each denominator, obtain $2x + 12 - 104 = 5x + 8$. Hence the

RULE.

Multiply each term of the equation by the least common multiple of the denominators, and reduce fractional to entire terms.

NOTE 1. Also, an equation may be cleared of fractions by multiplying each numerator by all the denominators except its own.

NOTE 2. It must be observed, that when a fraction is preceded by —, the sign requires the value of the fraction to be subtracted, so that, on removing its denominator, all the signs of its numerator must be changed. (Art. 55.)

NOTE 3. A negative exponent is used to indicate that the single letter or quantity to which it is attached is a divisor, or denominator (Art. 71); hence, negative exponents are removed from an equation in the same way as fractions. To clear an equation of a negative exponent, we should then multiply each of its terms by the letter or quantity which bears that negative exponent, the multiplier taking an equal positive exponent. If there are two or more letters which have negative exponents, we should multiply by their least common multiple, with the signs of the exponents changed.

Explain the operation. Repeat the Rule. What is Note 1? Note 2? Note 3?

Clear the equations of their fractions in the following

EXAMPLES.

2. $\frac{x}{a} = b + c.$ Ans. $x = ab + ac.$

3. $x + \frac{x}{2} + \frac{x}{4} = 7.$ Ans. $4x + 2x + x = 28.$

4. $x - \frac{x}{3} - 13 = \frac{x}{2}.$ Ans. $6x - 2x - 78 = 3x.$

5. $x + \frac{x}{2} - 40 = \frac{7x}{10}.$ Ans. $10x + 5x - 400 = 7x.$

6. $a + \frac{1}{x} = b + c + \frac{d}{x}.$
Ans. $ax + 1 = bx + cx + d.$

7. $m + x^{-1} = n - px^{-1}.$ Ans. $mx + 1 = nx - p.$

8. $a - \frac{1}{2} + b = c + \frac{d}{2} - x.$

9. $x - \frac{4x + 8}{6} = 8.$ Ans. $6x - 4x - 8 = 48.$

10. $\frac{22x + 42}{3x + 7} = 7.$ Ans. $22x + 42 = 21x + 49.$

11. $a + x = \frac{x^2 + 2ab}{a + x}.$
Ans. $a^2 + 2ax + x^2 = x^2 + 2ab.$

12. $\frac{43 - 7y}{5} = \frac{69 - 9y}{11}.$
Ans. $473 - 77y = 345 - 45y.$

13. $x + \frac{3x - 5}{2} = 12 - \frac{2x - 4}{3}.$
Ans. $6x + 9x - 15 = 72 - 4x + 8.$

14. $\frac{a - x}{b} - \frac{4a - x}{c} = a - b^{-1}.$
Ans. $ac - cx - 4ab + bx = abc - c.$

15. $x - 12 = \frac{3}{8}(44 - x + 12).$
Ans. $8x - 96 = 132 - 3x + 36.$

16. $\frac{2x}{3} - 6\frac{2}{3} = \frac{4}{9}(30 - x)$.

Ans. $6x - 60 = 120 - 4x$.

17. $\frac{x-2}{x+2} + \frac{x+2}{x-2} = 14$.

Ans. $x^2 - 4x + 4 + x^2 + 4x + 4 = 14(x^2 - 4)$.

SOLUTION OF SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

154. The Solution of an Equation is the process of finding the value of the unknown quantity in the equation.

155. The Root of an equation is the value of its unknown quantity.

156. The root of an equation is *found* by bringing all the terms containing the unknown quantity into one member, and freeing it from all connection with known quantities.

157. The root of an equation is *verified*, or the equation *satisfied*, when, the root being substituted for its symbol in the equation, the members are found to be equal, and the equation is thus reduced to an identical one.

158. The unknown and the known terms of an equation may be combined in various ways:—

1. By addition; as, $x + 6 = 18$, or $x + a = b$.
2. By subtraction; as, $x - 3 = 10$, or $x - a = d$.
3. By multiplication; as, $3x = 24$, or $ax = c$.
4. By division; as, $\frac{x}{4} = 8$, or $\frac{x}{c} = a$.
5. By a combination of two or more of these; as,

$$\frac{3x}{2} + 10 = 2x - 5, \text{ or } \frac{ax}{b} + e = cx - d.$$

What is meant by the Solution of an equation? Root of an equation? How is it found? When verified? How may the unknown and the known terms of an equation be combined?

159. To solve simple equations of one unknown quantity.

1. In the equation $x + 9 = 20$, find the value of x.

OPERATION.

$$x + 9 = 20$$
$$x = 20 - 9$$
$$x = 11$$

VERIFICATION.

$$11 + 9 = 20$$
$$20 = 20$$

Transposing the known quantity in the first member to the second (Art. 152), we have $x = 20 - 9$; and uniting the terms of the second member of this equation, by performing the subtraction indicated, we obtain 11 as the value of x. This value of x we verify, and find it satisfies the equation (Art. 157).

2. In the equation $x - 7 = 11$, find the value of x.

OPERATION.

$$x - 7 = 11$$
$$x = 11 + 7$$
$$x = 18$$

VERIFICATION.

$$18 - 7 = 11$$
$$11 = 11$$

Transposing the known quantity in the first member to the second (Art. 152), and, in the equation obtained, performing the addition indicated, we have 18 as the value of x. This value of x we verify, and find it to satisfy the equation (Art. 157.)

3. In the equation $6x + 24 = 72$, find the value of x.

OPERATION.

$$6x + 24 = 72 \quad (1)$$
$$6x = 48 \quad (2)$$
$$x = 8 \quad (3)$$

VERIFICATION.

$$48 + 24 = 72 \quad (1)$$
$$72 = 72 \quad (2)$$

Transposing 24, and uniting the known terms by subtraction, we have (2); and dividing both members of (2) by 6, the coefficient of x, we obtain 8 as the value of x.

This value of x being substituted in the original equation, and the terms of the first member united by addition, we have (2), an identical equation; therefore, the value is verified and the equation satisfied.

Explain the first operation. The second. The third.

4. In the equation $x - \frac{x}{2} + \frac{3x}{4} = 20 + 5$, find the value of x.

OPERATION.

$$
\begin{aligned}
x - \frac{x}{2} + \frac{3x}{4} &= 20 + 5 & (1)\\
4x - 2x + 3x &= 80 + 20 & (2)\\
5x &= 100 & (3)\\
x &= 20 & (4)
\end{aligned}
$$

Clearing the equation of fractions (Art. 153), we obtain (2); uniting similar terms, we have (3); and dividing both members of (3) by 5, the coefficient of x, we obtain for the value of x, 20.

VERIFICATION.

$$
\begin{aligned}
20 - \frac{20}{2} + \frac{60}{4} &= 20 + 5\\
20 - 10 + 15 &= 20 + 5\\
25 &= 25
\end{aligned}
$$

This value, by verification, we find satisfies the given equation.

From the preceding operations, it will be noticed that, when the unknown quantity is combined with known quantities by *addition* or *subtraction*, it may be separated by *transposition;* when combined by *multiplication*, it may be separated by *division;* and when combined by *division*, it may be separated by *multiplication*.

It will be observed that the first and last of these cases have been fully treated under the heads of *transposition* (Art. 152), and *clearing of fractions* (Art. 153). The second case most frequently occurs as the last step in the solution of an equation, when the coefficient of the unknown quantity is to be removed by division. It is usually, therefore, very simple, and has not been treated under a separate head. If at any time, however, all the terms of an equation can be exactly divided by any quantity, the equation may be thus simplified.

Combining the principles illustrated by the foregoing examples, we have, for the solution of simple equations containing only one unknown quantity, the following general

---

Explain the the fourth operation. How is the unknown quantity separated from known quantities?

RULE.

Clear the equation of fractions, if it has any.

Transpose the unknown terms to the first member, and the known terms to the second member, and reduce each member to its simplest form.

Divide both members by the coefficient of the unknown quantity, and the second member of the resulting equation will be the value of the unknown quantity.

NOTE. If the coefficient of the unknown quantity is negative, in dividing it must be remembered that like signs produce + and unlike signs produce — (Art. 67). Thus, $-3x = -24$, divided by -3, the coefficient of x, becomes $x = 8$.

The negative sign may also be removed by changing the signs of all the terms of the equation (Art. 152, Note). The positive coefficient would then be used as a divisor.

EXAMPLES.

5. Given $5x + 43 - 5 = 100 - 27$, to find x.
Ans. $x = 7$.

6. Given $7x + 7 + 1 = 96 - 11$, to find x.
Ans. $x = 11$.

7. Given $15x + 8 - 9 = 212 + 87$, to find x.
Ans. $x = 20$.

8. Given $9x + 9 = x - 71$, to find x.
Ans. $x = -10$.

9. Given $4x - 15 = 2x + 13$, to find x.
Ans. $x = 14$.

10. Given $4(x - 12) = 2(12 - x)$, to find x.
Ans. $x = 12$.

NOTE. Performing the multiplication indicated, the given equation becomes $4x - 48 = 24 - 2x$.

Repeat the Rule. The Note.

11. Given $9(x+1)=12(x-2)$, to find x.
Ans. $x=11$.

12. Given $3(x-3)+2x=3(40-x-19)$, to find x.
Ans. $x=9$.

13. Given $3(2x+3x)-15=72-4(x-2)$, to find x.
Ans. $x=5$.

14. Given $2(x-6)+3(2x+5)=3(3x-2)-1$, to find x.
Ans. $x=10$.

15. Given $x-\frac{x}{2}-\frac{x}{6}=30$, to find x. Ans. $x=90$.

16. Given $1-8x^{-1}=\frac{1}{5}+8x^{-1}$, to find x.
Ans. $x=20$.

Note. We may either free the equation of its negative exponents, or find the value of x^{-1} and take its reciprocal (Art. 71).

17. Given $\frac{2}{3}x+12=\frac{4}{5}x+6$, to find x.
Ans. $x=45$.

18. Given $\frac{x}{3}+\frac{3x}{5}+\frac{4x}{7}=158$, to find x.
Ans. $x=105$.

19. Given $\frac{x}{2}+\frac{x}{4}+\frac{x}{8}=28$, to find x. Ans. $x=32$.

20. Given $ax+b=cx+d$, to find x.

OPERATION.

$$ax+b=cx+d \quad (1)$$
$$ax-cx=d-b \quad (2)$$
$$(a-c)x=d-b \quad (3)$$
$$x=\frac{d-b}{a-c} \quad (4)$$

Transposing, we obtain (2); factoring the first member of (2), we have (3); and dividing by $a-c$, the coefficient of x, we obtain for the value of x, $\frac{d-b}{a-c}$.

21. Given $\frac{ax}{n}=d$, to find x. Ans. $x=\frac{nd}{a}$.

22. Given $\frac{ax}{2}+\frac{bx}{3}=c$, to find x.
Ans. $x=\frac{6c}{3a+2b}$.

Explain the operation of Example 20.

23. Given $\frac{1}{a}+\frac{b}{x}=c$, to find x. Ans. $x=\frac{a\,b}{a\,c-1}$.

24. Given $x+n\,x=a$, to find x. Ans. $x=\frac{a}{1+n}$.

25. Given $a-c\,x^{-1}=d\,x^{-1}-b$, to find x.

Ans. $x=\frac{c+d}{a+b}$.

26. Given $\frac{b\,x}{a}-\frac{d}{c}=\frac{a}{b}-\frac{c\,x}{d}$, to find x.

Ans. $x=\frac{a\,d}{b\,c}$.

NOTE. Reduce the value of x to its lowest terms, by casting out the common binomial factor.

27. Given $\frac{2\,x+1}{3}=x-\frac{x+3}{4}$, to find x.

Ans. $x=13$.

28. Given $\frac{3\,x+4}{3}-\frac{24-x}{3}=10-2\,x$, to find x.

OPERATION.

$$\frac{3\,x+4}{3}-\frac{24-x}{3}=10-2\,x$$
$$3\,x+4-24+x=30-6\,x$$
$$10\,x=50$$
$$x=5$$

The second fraction being preceded by —, on removing the denominator, we change all the signs of the numerator. (Art. 153, Note 2.)

29. Given $5-\frac{x+4}{11}=x-3$, to find x.

Ans. $x=7$.

30. Given $2\,x+\frac{24}{4}=\frac{3\,x}{4}-4$, to find x.

Ans. $x=-8$.

31. Given $x+\frac{x+8}{4}-\frac{x-6}{3}=x-2$, to find x.

Ans. $x=72$.

Explain the operation of Example 28.

32. Given $\frac{3x}{4} + \frac{x-4}{2} - \frac{4x-40}{8} = x - 6$, to find x.
Ans. $x = 36$.

33. Given $\frac{5x-11}{4} - \frac{x-1}{10} = \frac{11x-1}{12}$, to find x.
Ans. $x = 11$.

34. Given $\frac{4x+3}{9} + \frac{7x-29}{5x-12} = \frac{8x+19}{18}$, to find x.

OPERATION.

$$\frac{4x+3}{9} + \frac{7x-29}{5x-12} = \frac{8x+19}{18} \qquad (1)$$

$$8x + 6 + \frac{126x-522}{5x-12} = 8x + 19 \qquad (2)$$

$$\frac{126x-522}{5x-12} = 13 \qquad (3)$$

$$126x - 522 = 65x - 156 \qquad (4)$$

$$61x = 366 \qquad (5)$$

$$x = 6. \qquad (6)$$

We multiply (1) by 18, one of the denominators, and obtain (2); and subtracting $8x + 6$ from each member (Art. 151), have (3). In like manner, operations may often be much abbreviated by multiplying first by the more simple of the denominators; and also, by reducing each resulting equation as much as possible.

35. Given $\frac{7x+16}{21} - \frac{x+8}{4x-11} = \frac{x}{3}$, to find x.
Ans. $x = 8$.

36. Given $\frac{5x-2}{8} + \frac{3x+22}{2(x+3)} = \frac{5x+14}{8}$, to find x.
Ans. $x = 10$.

37. Given $\frac{4x+17}{9} - \frac{26x-4}{17x+32} + \frac{2x}{3} = \frac{14x}{12} - \frac{2x-32}{36}$, to find x.
Ans. $x = 4$.

38. Given $\frac{13-x}{2} = \frac{6(x-5)}{8} + \frac{8x+15}{6}$, to find x.
Ans. $x = 3$.

NOTE. The equation can be multiplied by 2, by dividing each of the denominators by 2, the denominator of the first fraction.

39. Given $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$, to find x.
Ans. $x=4$.

40. Given $\frac{2x}{5}+\frac{3x+5}{5x-25}=\frac{6x+13}{15}$, to find x.
Ans. $x=20$.

41. Given $\frac{x}{1+a}+\frac{x}{1-a}=b$, to find x.
Ans. $x=\frac{b}{2}(1-a^2)$.

42. Given $x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}$, to find x.
Ans. $x=6$.

43. Given $\frac{ax}{b}-e=\frac{mx}{n}+d$, to find x.
Ans. $x=\frac{ben+bdn}{an-bm}$.

44. Given $\frac{3a+x}{x}-5-\frac{6}{x}=0$, to find x.
Ans. $x=\frac{3a-6}{4}$.

45. Given $ax^2+b^2x=a^2x+bx^2$, to find x.
Ans. $x=a+b$.

Note. First reduce the equation to the first degree.

46. Given $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$, to find x.
Ans. $x=\frac{ac}{b}$.

Note. $\frac{ac}{b}$ is the answer in its simplest form.

47. Given $\frac{x}{a-b}-\frac{2+x}{a+b}=\frac{c}{a^2-b^2}$, to find x.
Ans. $x=\frac{c+2a-2b}{2b}$.

48. Given $ax-\frac{a^2-3bx}{a}-ab^2=bx+\frac{6bx-5a^2}{2a}-\frac{bx+4a}{4}$, to find x.
Ans. $x=\frac{2a(2b^2-5)}{4a-3b}$.

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

160. The Solution of a Problem by Algebra, as has been already shown (Art. 44), consists of two distinct parts: —

1. The *Statement* of the problem in algebraic language.
2. The *Solution*, which determines the values of the unknown quantities.

The Statement is usually in the form of an equation, and the Solution is, then, that of the equation.

161. Problems often include in their solution the consideration of ratio and proportion, especially in expressing relations of algebraic quantities.

162. Ratio is the relation, in respect to magnitude, which one quantity bears to another of the same kind; or the quotient arising from the division of one quantity by another.

Thus, $\frac{a}{b}$ is the ratio of a to b. Ratio may be written in the form of a fraction, as $\frac{a}{b}$, or with two dots (:) between the two terms, as $a : b$, to be read a is to b.

163. Proportion is an equality of ratios.

Thus, $4 : 2 = 6 : 3$, or $a : b = c : d$, is a proportion. It may be written either with the sign of equality ($=$), or with four dots (: :), between the ratios; as $4 : 2 :: 6 : 3$, to be read 4 is to 2 as 6 is to 3.

164. Any four quantities, then, are said to be *proportional* to each other, when the first contains the second as many times as the third contains the fourth.

Of what parts does the Solution of a Problem consist? Define Ratio. Proportion. When are any four quantities said to be proportional to each other?

Thus, 9, 3, 12, and 4 are proportional, since 9 contains 3 as many times as 12 contains 4.

165. The first and last terms of a proportion are called EXTREMES, and the middle terms MEANS.

Thus, in $a : b :: c : d$, a and d are the extremes, and b and c the means.

166. In any proportion, the product of the extremes is equal to the product of the means.

Let $a : b :: c : d$; then $a \times d = b \times c$.

For, since the quantities are in proportion,

$$\frac{a}{b} = \frac{c}{d};$$

and clearing of fractions, $ad = bc$.

But ad is the product of the extremes, and bc the product of the means. Hence,

To convert a proportion into an equation, place the product of the extremes equal to the product of the means.

Thus, $x : 16 :: 20 : 4$ may be converted into the equation $4x = 320$.

167. It is impossible to give any general or precise rule for stating or solving every problem; yet the following directions may furnish some aid.

1. *Denote the unknown quantity or quantities by some of the final letters of the alphabet.*

2. *Form an equation, by indicating the operations required to verify the answer, were it already obtained.*

3. *Determine the value of the unknown quantity in the equation thus formed.*

What terms are called the extremes? What the means? Show that the product of the extremes is equal to the product of the means. How is a proportion converted into an equation? What directions are given for solving problems?

PROBLEMS.

1. There are two numbers, whose differenc whose sum is 43; what are the numbers?

SOLUTION.

Let $x =$ the smaller number,
and $x + 9 =$ the larger number.
Their sum, $x + x + 9 = 43$
Transposing and uniting, $2x = 34$
Dividing by 2, $x = 17$, the smaller number.
Then, $x + 9 = 26$, the larger number.

VERIFICATION.

$$26 - 17 = 9, \text{ and } 26 + 17 = 43.$$

2. It is required to find two numbers whose sum shall be 40 and their difference 16. Ans. 12 and 28.

3. At a certain election, 1296 persons voted, for two candidates, and the successful candidate had a majority of 120; how many voted for each? Ans. 588 and 708.

4. Find two numbers whose difference is 13, and which are such that if 17 be added to their sum, the whole will amount to 62. Ans. 16 and 29.

5. A bankrupt owes B twice as much as he owes A, and C as much as he owes A and B together; out of $3000, which is to be divided among them, what should each receive? Ans. A, $500; B, $1000; and C, $1500.

6. A company of 266 persons consists of men, women, and children; there are four times as many men as children, and twice as many women as children. How many are there of each?

Ans. 38 children, 76 women, 152 men.

Explain the solution of Problem 1.

7. Two trains of cars start at the same time towards each other, the one from Albany, running 26 miles per hour, and the other from Boston, 24 miles per hour; in what time will they meet, the distance by railroad being supposed to be 200 miles?

SOLUTION.

Let $x =$ number of hours required.

Then $26x =$ distance run by one,

and $24x =$ distance run by the other.

Their sum, $26x + 24x = 200$

Or, $50x = 200$

Whence, $x = 4$, number of hours required.

VERIFICATION.

$$26 \times 4 + 24 \times 4 = 200.$$

8. If two persons start at the same time from places 396 miles apart, and travel towards each other, the one at the rate of 36 miles per day, and the other 30 miles per day, in how many days will they meet, and how far will each have traveled?

Ans. In 6 days; the one will have traveled 216 miles, the other 180 miles.

9. A person starts from a certain place, and travels at the rate of 4 miles per hour; after he has been traveling 10 hours, a horseman, riding 9 miles per hour, is despatched after him; how many hours must the horseman ride to overtake him? Ans. 8 hours.

10. A house and garden cost $ 850, and five times the price of the house was equal to twelve times the price of the garden; find the price of each.

Ans. House, $ 600; garden, $ 250.

11. Two shepherds owning a flock of sheep agree to

Explain the solution of Problem 7.

divide its value equally; A takes 72 sheep, and B takes 92 sheep and pays A $ 35. Required the value of a sheep. Ans. $ 3.50.

12. Divide a line 21 inches long into two parts, such that one may be three fourths of the other.

SOLUTION.

Let $x =$ length of one part,

and $\frac{3x}{4} =$ length of the other part.

Then, $x + \frac{3x}{4} = 21$

Clearing of fractions, $4x + 3x = 84$

Or, $7x = 84$

Whence, $x = 12$, length of one part.

Then, $\frac{3x}{4} = 9$, length of the other part.

13. John's age is once and three fifths the age of James, and the sum of their ages is 39 years; required the age of each.

Ans. John's, 24 years; James's, 15 years.

14. A, B, and C have altogether $ 145; A's share is two thirds, and B's three fourths, as great as C's; what is the share of each?

Ans. A's, $ 40; B's, $ 45; C's, $ 60.

15. A man being asked his age, replied that, if it were increased by a half and a third of itself, it would be 44 years; what was his age? Ans. 24 years.

16. A person spends one fourth of his yearly income in board, and one seventh in other expenses, and saves $ 85; what is his income?

Explain the solution of Problem 12.

SOLUTION.

Let $28x =$ the number of dollars of income.
Then $\frac{1}{4}$ of $28x = 7x =$ what he spends in board,
and $\frac{1}{7}$ of $28x = 4x =$ what he spends in other expenses.
Then, $7x + 4x + 85 = 28x$
Or, $-17x = -85$
Whence, $x = 5$
Then, $28x = 140$, number of dollars of income.

To avoid fractions, we resort to the artifice of supposing $28x$ to be the number of dollars of his yearly income, 28 being chosen because it is divisible by both 4 and 7, the denominators of the given fractions; then, by the question, he spends in board $7x$ dollars, and in other expenses $4x$ dollars, and $7x + 4x + 85$ equal $28x$, or the yearly income. Thus, when fractions are foreseen to enter an equation, it will often be better to use, instead of x, such a multiple of x as will preclude their entrance.

17. There is a pole standing one half and one third of its length under water, and 4 feet above; required the length of the pole. Ans. 24 feet.

18. A man having completed two fifths of a journey, finds that, after traveling 30 miles farther, only three sevenths of the journey remain; required the length of the journey. Ans. 175 miles.

19. From a cask one third full of oil, there leaked out 21 gallons, when there was found to be just half the oil left; required the capacity of the cask.

Ans. 126 gallons.

20. There are three brothers whose ages together amount to 24 years, and their birthdays are two years apart. What is the age of each?

Ans. Youngest, 6 years; next, 8 years; oldest, 10 years.

21. A and B have together a dollars, but B's share is n times as great as A's; what is each one's share?

Explain the solution of Problem 16.

SOLUTION.

Let $x =$ A's share,

and $nx =$ B's share.

Their sum, $x + nx = a$

Or, $(1 + n)x = a$

Whence, $x = \frac{a}{1 + n}$, A's share.

Then, $nx = \frac{na}{1 + n}$, B's share.

22. A man bought the same number of pounds each of coffee at a cents, tea at b cents, and sugar at c cents per pound, and the whole amounted to d cents; required the number of pounds of each. Ans. $\frac{d}{a + b + c}$.

23. Twice my age, increased by b, is equal to a; what is my age? Ans. $\frac{a - b}{2}$ years.

24. At a certain election, a persons voted, and the successful candidate had a majority of b; how many votes did he receive? Ans. $\frac{a + b}{2}$.

25. My carriage is worth $1\frac{1}{2}$ times as much as my horse, and both together are worth c dollars; what is the value of each? Ans. Horse, $\frac{2c}{5}$; carriage, $\frac{3c}{5}$.

26. A courier left this place n days ago, and goes a miles each day. He is pursued by another who goes b miles daily. In how many days will the second, starting to-day, overtake the first? Ans. $\frac{na}{b - a}$ days.

27. I have a certain number in my mind. I multiply it by 7, add 3 to the product, and divide the sum by 2; I then find that if I subtract 4 from the quotient, I get 15; what number am I thinking of? Ans. 5.

Explain the solution of Problem 21.

28. From one end of a rod is cut away a fifth part of it, and from the other end 3 inches more than a sixth part, and there remains 16 inches; required the length of the rod. Ans. 30 inches.

29. A and B had equal sums of money; A lost $ 50 more than a quarter of his, and B gained as much as A lost; then B had twice as much as A; what sum had each at first?

SOLUTION.

Let $x =$ what each had at first.

Then $x - \frac{x}{4} - 50 =$ what A had after losing,

and $x + \frac{x}{4} + 50 =$ what B had after gaining.

Then, $x + \frac{x}{4} + 50 = 2\left(x - \frac{x}{4} - 50\right)$

Or, $x + \frac{x}{4} + 50 = 2x - \frac{x}{2} - 100$

Or, $-x + \frac{x}{4} + \frac{x}{2} = -150$

Whence, $-x = -600$

Or, $x = 600$, what each had at first.

30. I buy four houses; for the second I give half as much again as for the first, for the third half as much again as for the second, and for the fourth as much as for the first and third together; I pay for the whole $ 8000. What is the cost of each?

Ans. First, $ 1000; second, $ 1500; third, $ 2250; and fourth, $ 3250.

31. A father is three times as old as his son, but five years ago he was four times as old; what are their ages now? Ans. Son's age, 15 years; father's, 45 years.

32. A vessel holding 120 gallons is partly filled by a

Explain the solution of Problem 29.

spout which delivers 14 gallons in a minute; this is then turned off, and a second spout, delivering 9 gallons in a minute, completes the filling of the vessel. How long did each spout run, the time occupied by both being 10 minutes?

Ans. The first, 6 minutes; the second, 4 minutes.

33. A can do a piece of work in a days, which it requires b days for B to perform; in how many days can it be done if A and B work together?

SOLUTION.

Let x = the number of days required.

Then $\frac{1}{a}$ = what A can do in one day,

and $\frac{x}{a}$ = what A can do in x days.

Also $\frac{1}{b}$ = what B can do in one day,

and $\frac{x}{b}$ = what B can do in x days.

Then, $\frac{x}{b} + \frac{x}{a} = 1$

Clearing of fractions, $ax + bx = ab$

Or, $(a+b)x = ab$

Whence, $x = \frac{ab}{a+b}$, number of days required.

Let x be the number of days, and 1 the entire work; then, in 1 day A can do $\frac{1}{a}$ of the work, and B $\frac{1}{b}$, therefore, in x days, they can do $\frac{x}{a}$ and $\frac{x}{b}$ of the work. Hence, by the conditions of the question, $\frac{x}{b} + \frac{x}{a} = 1$.

34. A can mow a field in 3 days, which it takes B

Explain the solution of Problem 33.

7 days to mow; in how many days can it be mown by A and B working together? Ans. $2\frac{1}{10}$ days.

As a and b may have any value whatever, and retain their identity in the final result, the solution of Problem 33 furnishes a formula which can be used for the solution of any similar problem. Thus, to obtain the required result in Problem 34, we have only to substitute 3 for a and 7 for b, which gives

$$x = \frac{ab}{a+b} = \frac{21}{10} = 2\tfrac{1}{10}.$$

A problem is said to be *generalized* when letters are, in this manner, used to represent its known quantities.

The above formula may be expressed as an arithmetical rule, thus: When the times are known in which two agencies, acting separately, can accomplish a certain result, the time required for them conjointly to accomplish the same result may be found by *dividing the product of the given times by their sum.*

The principle demonstrated by any other general problem may be drawn from the formula in a similar manner.

35. A can perform a piece of work in a days, B in b days, and C in c days; in how many days will they accomplish the work, if they all work together?

Ans. $\dfrac{abc}{ab+ac+bc}$ days.

It will be seen that, when three agencies are employed, the required time is the product of the given times, divided by the sum of their products, taken two and two.

36. A cistern can be filled by three pipes; by the first in 2 hours, by the second in 3, and by the third in 4; in what time can it be filled by all the pipes running together? Ans. 55 min. $23\frac{1}{13}$ sec.

37. How many pounds of sugar at 9 cents a pound must be mixed with 20 pounds at 13 cents, in order that the mixture may be worth 10 cents a pound?

When is a Problem said to be generalized?

SOLUTION.

Let $x =$ number of pounds at 9 cents,
and $x + 20 =$ number of pounds in the mixture.
Then $9x =$ value of x pounds at 9 cents,
and $10x + 200 =$ value of $x + 20$ pounds at 10 cents.
Also $260 =$ value of 20 pounds at 13 cents.

Then, $9x + 260 = 10x + 200$
Whence, $-x = -60$
Or, $x = 60$, number of pounds at 9 cents.

38. How much rye at four shillings and sixpence a bushel must be mixed with fifty bushels of wheat at six shillings a bushel, that the mixture may be worth five shillings a bushel? Ans. 100 bushels.

39. A liquor agent has 40 gallons of superior wine, worth $7 a gallon; he wishes, however, so to reduce its quality, by the addition of water, that he may sell it at $4.50 a gallon; how much water must he add?
Ans. $22\frac{2}{9}$ gallons.

40. A banker lets three fifths of his money at 5 per cent, and the remainder at 6 per cent, and at the end of the year receives $1080 interest. What is the amount let?

SOLUTION.

Let $5x =$ amount let.
Then $3x =$ amount at 5 per cent,
and $2x =$ amount at 6 per cent.

Then, $3x \times \frac{5}{100} + 2x \times \frac{6}{100} = 1080$
Or, $\frac{15x}{100} + \frac{12x}{100} = 1080$
Clearing of fractions, $15x + 12x = 108000$
Or, $27x = 108000$
Whence, $x = 4000$
and $5x = 20000$, amount let.

Explain the solution of Problem 37. Problem 40.

41. A capitalist has two thirds of his money in United States 6 per cent stocks, and the balance in 8 per cent railroad bonds; his yearly income from both is $1200; required the amount in each investment.

Ans. In United States stocks $12000, and in railroad bonds $6000.

42. The rent of an estate this year is $1890, which is 8 per cent greater than it was last year; what was it last year? Ans. $1750.

43. A merchant adds yearly to his capital 40 per cent of it, but takes from it, at the end of each year, $3000 for expenses. After deducting the last $3000, at the end of the second year, he finds his original capital has been increased 60 per cent. What was that capital?

Ans. $20000.

44. Of my income, $\frac{1}{4}$ is derived from bank-stock, $\frac{1}{5}$ from a farm, $\frac{1}{2}$ from a factory, and the aggregate from these sources is $3800. Required my entire income.

SOLUTION.

Let $x =$ the entire income,

and $a = 3800.$

Then, $\frac{x}{5} + \frac{x}{4} + \frac{x}{2} = a$

Clearing of fractions, $4x + 5x + 10x = 20a$

Or, $19x = 20a$

Whence, $x = 1\frac{1}{19}a$

Or, $x = 4000$, the entire income.

We here represent the numeral 3800 by a letter, and in the result restore its value. An artifice of this kind may often be advantageously used, in order to avoid the use of large numbers.

Explain the solution of Problem 44.

45. A young man, by putting three sevenths of his earnings in the savings' bank and one eighth into government stocks, found at the end of the year that he had thus laid by $ 930. Required the amount of his yearly earnings.

Ans. $ 1680.

46. Divide the number a into two parts that shall have to each other the ratio of m to n.

FIRST SOLUTION.

Let $x =$ one part,

and $a - x =$ the other part.

Then, $x : a - x = m : n$

Whence, $nx = ma - mx$

Or, $mx + nx = ma$

And $(m + n)x = ma$

Whence, $x = \frac{ma}{m+n}$, one part,

and $a - x = \frac{na}{m+n}$, the other part.

SECOND SOLUTION.

Let $mx =$ one part,

and $nx =$ the other part.

Then, $mx + nx = a$

Or, $(m + n)x = a$

Whence, $x = \frac{a}{m+n}$

$mx = \frac{ma}{m+n}$, one part,

and $nx = \frac{na}{m+n}$, the other part.

47. Divide 34 into two such parts, that the difference between the greater and 18 shall be to the difference between 18 and the less in the proportion of 2 to 3.

Ans. 22 and 12.

Explain the solution of Problem 46.

48. A person has 264 coins, dollars and eagles; the number of dollar pieces is to the number of eagles in the ratio of 9 to 2; how many of each coin has he?

Ans. Dollar pieces, 216; eagles, 48.

49. The ages of two persons are in the ratio of 3 to 4, but 5 years ago the ratio of their ages was that of 2 to 3; what are their ages? Ans. 15 and 20.

50. Two pieces of cloth were purchased at the same price per yard, but as they were of different lengths, the one cost $5, and the other $6.50. If each had been 10 yards longer, their lengths would have been as 5 to 6. Required the length of each piece.

Ans. 20 and 26.

51. A market woman bought some eggs at 2 for a cent, and as many more at 3 for a cent; she sold them all at the rate of 5 for 2 cents, and found she had lost 4 cents. How many did she buy of each sort?

SOLUTION.

Let $x =$ the number of each sort.

Then $\frac{x}{2} =$ the cost of the first sort,

and $\frac{x}{3} =$ the cost of the second sort.

But $2x =$ the entire number,

and $2x \times \frac{2}{5} = \frac{4x}{5} =$ amount received for whole.

Then, $\frac{x}{2} + \frac{x}{3} - \frac{4x}{5} = 4$

Clearing of fractions, $15x + 10x - 24x = 120$

Whence, $x = 120$, number of each sort.

52. Two merchants, A and B, traded in company, with a joint stock of $6300. A's money was employed 12

Explain the solution of Problem 51.

months, and B's 8 months; and, on dividing profits, each had gained exactly the same sum. How much capital did each furnish? Ans. A, $2520; B, $3780.

53. A workman was employed for 60 hours, on condition that for every hour he worked he should receive 15 cents, and for every hour he was idle he should forfeit 5 cents; at the end of the time he received $2.40. Required the number of hours he worked, and the number he was idle.

SOLUTION.

Let $x =$ number of hours he worked,

and $60 - x =$ number of hours he was idle.

Then $15x =$ his pay for working,

and $5(60 - x) =$ his forfeiture for being idle.

Then, $15x - 5(60 - x) = 240$

Or, $15x - 300 + 5x = 240$

Whence, $20x = 540$

And $x = 27$, number of hours he worked.

Then, $60 - x = 33$, number of hours he was idle.

54. A workman engaged for 48 days at the rate of $2 per day and his board, which is estimated at $1 per day. At the end of the time he receives $42 only, his employer having deducted the cost of his board for every day he was idle. How many days did he work? Ans. 30 days.

55. Two casks contain equal quantities of vinegar; from the first 34 quarts are drawn, and from the second 80; the quantity remaining in one vessel is now twice that in the other. How much did each cask originally contain? Ans. 126 quarts.

Explain the solution of Problem 53.

56. Two thirds of a certain number of persons received 18 cents each, and one third received 30 cents each. The whole sum received was $6.60. How many persons were there? Ans. 30.

57. There is a fish whose head weighs 12 pounds, his tail weighs as much as his head and half the weight of his body, and his body weighs 26 pounds more than his head and tail both. Required the weight of the fish.
Ans. 174 pounds.

58. A boatman who can row at the rate of 9 miles an hour, finds that it takes twice as long to row his boat up river a certain distance, as to row it down river the same distance; at what rate does the river flow?
Ans. 3 miles per hour.

59. The paving of a square court with stone, at 40 cents a yard, will cost as much as the enclosing it with an iron fence, at $1 a yard; what is the length of the side of the square in yards?

SOLUTION.

Let $x =$ length of the side in yards.
Then $4x =$ number of yards of fence,
and $x^2 =$ number of yards of pavement.
Hence $4x \times 100 = 400 =$ cost of fence,
and $\underline{x^2 \times 40 = 40x^2 =}$ cost of paving.
Then, $40x^2 = 400x$
Dividing by x, $40x = 400$
Whence, $x = 10$, length of the side in yards.

60. A farmer has hogs worth $12.50, and pigs worth $2.50, each; the number of hogs and pigs being 35, and their value $197.50. Required the number he has of each. Ans. Hogs, 11; pigs, 24.

Explain the solution of Problem 59.

61. A lady being asked her age, replied, that if a half of her age were taken from it, and also a half of that remainder were taken away, she should be 19. Required her age. Ans. 76 years.

62. A gentleman hired a servant for 12 months, at the wages of $90 and a suit of clothes. At the end of 7 months, the man quits his service and receives $33.75 and the suit of clothes. At what price were the clothes estimated? Ans. $45.

63. A gentleman having $12000, employs a portion of the money in building a house. One third of the money that remains he invests at 4 per cent, and the other two thirds at 5 per cent; from these two investments he obtains an income of $392. What was the cost of the house? Ans. $3600.

64. A person desirous of giving some children 3 cents apiece, found he had not money enough in his pocket by 8 cents; he therefore gave them each 2 cents, and had then 3 cents remaining; required the number of children. Ans. 11.

65. Three towns, A, B, and C, raise a sum of $11800; for every $20 which A contributes, B contributes $12, and C $18. What does each contribute?
Ans. A, $4720; B, $2832; C, $4248.

66. A newsboy gains during one day as much money as he had in the morning, but spends 16 cents at night; the next day he gains as much as he had that morning, and spends 16 cents at night; and so on, each day doubling his money, but spending 16 cents at night. At the close of the fourth evening, he finds that he has nothing left; how much money had he at first? Ans. 15 cents.

67. What number is that to which, if we add its fourth, fifth, and eighth, the sum will be 126? Ans. 80.

68. A and B find a purse containing gold dollars. A takes out two dollars and one sixth of what remains, then B takes out three dollars and one sixth of what remains, and they find that they have taken out equal shares; how many dollars were in the purse, and how much did each take out?

Ans. 20 dollars in the purse; 5 dollars taken out by each.

69. If from three times a certain number we subtract 8, half the remainder will be equal to the number itself diminished by 2; required the number.

70. A man and his wife usually consumed a bag of flour in 12 days; but when the man was from home, it lasted his wife 30 days; how many days would it last the man alone? Ans. 20 days.

71. At 12 o'clock both hands of a clock are together; when will they next be together?

Ans. At $5\frac{5}{11}$ minutes past 1 o'clock.

72. I have 90 sheep. If I would divide them into flocks, such that, if the number in the first be increased by 2, the number in the second diminished by 2, the number in the third multiplied by 2, and the number in the fourth divided by 2, the results will all be equal; how many must I put in each flock?

Ans. In the first 18, second 22, third 10, and fourth 40.

73. A certain article of consumption was subject to a duty of 6 cents a pound; in consequence of a reduction in the duty, the consumption increased one half, but the revenue fell one third; what was the duty on a pound after the reduction? Ans. $2\frac{2}{3}$ cents.

74. A hare is 50 of her own leaps before a greyhound, and takes 4 leaps to the greyhound's 3, but 2 of the greyhound's leaps are equal to 3 of the hare's; how many leaps must the greyhound take to catch the hare?

Ans. 300.

75. A general arranging his men in the form of a solid square, finds he has 21 men over, but attempting to add one man to each side of the square, finds he wants 200 men to fill up the square; required the number of men.
Ans. 12121.

Let $x =$ the number of men on a side at first, then $x^2 + 21 =$ the whole number of men.

SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

168. INDEPENDENT EQUATIONS are such as cannot be made to assume the same form.

If they relate to the same problem, they must, therefore, express essentially different conditions of that problem. Thus, $4x + 10y = 72 + 4y$ and $5x + 9y = 108 - x$ are not independent equations, because each reduces to the form of $2x + 3y = 36$.

169. When a problem requires two or more unknown quantities to be determined, it is necessary that there should be as many independent equations as there are unknown quantities.

For, if we have an equation containing two unknown quantities, x and y, as

$$x - y = 1,$$

transposing y, we have

$$x = 1 + y. \qquad (1)$$

But the value of y is not known; consequently, from this equation alone, the value of x cannot be determined.

If, however, we have a second equation, as

$$x + y = 7,$$

or,

$$x = 7 - y, \qquad (2)$$

in which the value of x and y are the same as in the first, the sec-

---

What is an Independent Equation? Show that there should be as many independent equations as there are unknown quantities.

ond members of (1) and (2) being equal to the same quantity, x, and consequently equal to each other (Art. 38, Ax. 7), give

$$1 + y = 7 - y;$$

or, $$2y = 6.$$

Whence, $$y = 3.$$

Substituting 3, the value of y, for y in either equation (1) or equation (2), we obtain 4 as the value of x; and the values obtained for the two unknown quantities satisfy the two equations.

170. SIMULTANEOUS EQUATIONS are those in which the unknown quantities are satisfied by the same values.

Two unknown quantities require for their determination, as shown in the preceding Article, at least *two* independent, simultaneous equations.

When by means of these we cause one of the unknown quantities to disappear, we are said to *eliminate* it.

ELIMINATION.

171. ELIMINATION is the process of deducing, from two or more simultaneous equations having two or more unknown quantities, a single equation having only one unknown quantity.

There are three methods of elimination, and consequently as many cases:—

I. By comparison.
II. By substitution.
III. By addition or subtraction.

CASE I.

172. To eliminate by comparison.

1. Given $3x + 4y = 20$, and $4x - 2y = 12$, to find the values of x and y.

Define Simultaneous Equations. How many simultaneous equations are required to determine two unknown quantities? Define Elimination.

OPERATION.

$$3x + 4y = 20 \tag{1}$$

$$4x - 2y = 12 \tag{2}$$

$$x = \frac{20 - 4y}{3} \tag{3}$$

$$x = \frac{12 + 2y}{4} \tag{4}$$

$$\frac{12 + 2y}{4} = \frac{20 - 4y}{3} \tag{5}$$

$$36 + 6y = 80 - 16y \tag{6}$$

$$22y = 44 \tag{7}$$

$$y = 2 \tag{8}$$

$$x = \frac{20 - 8}{3} \tag{9}$$

$$x = 4 \tag{10}$$

By transposing $4y$ in equation (1) and dividing by 3, also transposing $2y$ in equation (2) and dividing by 4, we have (3) and (4), equations in which the value of x is expressed in terms of y. Then, since each of the two quantities $\frac{12 + 2y}{4}$ and $\frac{20 - 4y}{3}$ is equal to x, they are equal to each other (Art. 38, Ax. 7); and placing them equal the one to the other, we obtain (5), an equation with only one unknown quantity, y. Reducing, we have (8), or $y = 2$. Substituting, now, 2 for y in equation (3), and reducing, we have (10), or $x = 4$. Hence the

RULE.

Find an expression for the value of the same unknown quantity in each of the equations, and form a new equation, by placing these values equal to each other.

NOTE 1. The equation thus formed is solved as we would solve any equation containing one unknown quantity.

NOTE 2. The value of the remaining unknown quantity may be determined in the same way as that of the first, thus making two independent solutions, one for each unknown quantity. When, however, the value already determined is a simple number, it is best to substitute that value for its symbol in some one of the equations, and thus obtain the value of the remaining unknown quantity.

NOTE 3. It is usually most convenient to reduce each equation to its simplest form, by clearing of fractions, transposing and uniting, &c., before attempting to eliminate, by either method.

Explain the operation. Repeat the Rule. Note 1. Note 2. Note 3.

EXAMPLES.

2. Given $2x + 3y = 23$, and $5x - 2y = 10$, to find the values of x and y. Ans. $x = 4$; $y = 5$.

3. Given $4x + y = 34$, and $x + 4y = 16$, to find the values of x and y. Ans. $x = 8$; $y = 2$.

4. Given $5x - 3y = 9$, and $2x + 5y = 16$, to find the values of x and y. Ans. $x = 3$; $y = 2$.

5. Given $7x + 3y = 13$, and $5x + 2y = 9$, to find the values of x and y.

6. Given $8x - 7y = -15$, and $3y - 6x = -9$, to find the values of x and y. Ans. $x = 6$; $y = 9$.

7. Given $14x + 6y = 0$, and $6x - 46 = 4y$, to find the values of x and y. Ans. $x = 3$; $y = -7$.

8. Given $\frac{x}{2} + \frac{y}{3} = 7$, and $\frac{x}{3} + \frac{y}{2} = 8$, to find the values of x and y. Ans. $x = 6$; $y = 12$.

9. Given $x + 2y = 17$, and $3x - y = 2$, to find the values of x and y. Ans. $x = 3$; $y = 7$.

10. Given $\frac{x}{2} - y = 1$, and $x - \frac{y}{2} = 8$, to find the values of x and y. Ans. $x = 10$; $y = 4$.

11. Given $\frac{x+y}{10} + \frac{x-y}{2} = 0$, and $\frac{x+y}{5} + \frac{x-y}{2} = 1$, to find the values of x and y. Ans. $x = 4$; $y = 6$.

12. Given $\frac{x+8}{4} + 6y = 21$, and $\frac{y+6}{3} = 23 - 5x$, to find the values of x and y. Ans. $x = 4$; $y = 3$.

CASE II.

173. To eliminate by substitution.

1. Given $x + 2y = 17$, and $3x - y = 2$, to find the values of x and y.

OPERATION.

$$\begin{array}{rll}
x + 2y = & 17 & (1) \\
3x - y = & 2 & (2) \\
\hline
x = & 17 - 2y & (3) \\
3(17-2y) - y = & 2 & (4) \\
51 - 7y = & 2 & (5) \\
-7y = & -49 & (6) \\
y = & 7 & (7) \\
x = & 17 - 14 & (8) \\
x = & 3 & (9)
\end{array}$$

By transposing $2y$ in equation (1), we have equation (3), which gives the value of x expressed in terms of y. Substituting this value of x, or $17 - 2y$, for x in equation (2), we obtain (4), an equation with only one unknown quantity, y. Reducing, we have (7), or $y = 7$. Substituting, now, 7 for y in equation (3), and reducing, we have (9), or $x = 3$.

2. Given $2x + 5y = 23$, and $3x - 2y = 6$, to find the values of x and y.

OPERATION.

$$\begin{array}{rl}
2x + 5y = 23 & (1) \\
3x - 2y = 6 & (2) \\
\hline
x = \dfrac{6 + 2y}{3} & (3) \\
\dfrac{2(6 + 2y)}{3} + 5y = 23 & (4) \\
12 + 4y + 15y = 69 & (5) \\
19y = 57 & (6) \\
y = 3 & (7) \\
x = \dfrac{6 + 6}{3} & (8) \\
x = 4 & (9)
\end{array}$$

By transposing $2y$ in equation (2) and dividing by 3, we have (3), which gives for the value of x, $\frac{6 + 2y}{3}$. Substituting this value of x in equation (1), we have (4), an equation with only one unknown quantity, y; and reducing, we have (7), or $y = 3$. Substituting 3 for y in equation (3), and reducing, we have (9), or $x = 4$. Hence the

RULE.

Find an expression for the value of one of the unknown quantities, in either equation, and substitute this value in the place of the same unknown quantity in the other equation.

Note. This method may be advantageously used when either of the unknown quantities has 1 for a coefficient.

Explain the operation of Example 1. Of Example 2. What is the Rule? The Note?

EXAMPLES.

3. Given $x + 4y = 16$, and $4x + y = 34$, to find the values of x and y. Ans. $x = 8$; $y = 2$.

4. Given $x + 2y = 18$, and $2x - y = 1$, to find the values of x and y. Ans. $x = 4$; $y = 7$.

5. Given $x + y = 13$, and $x - y = 3$, to find the values of x and y. Ans. $x = 8$; $y = 5$.

6. Given $\frac{x}{2} - y = 1$, and $x - \frac{y}{2} = 8$, to find the values of x and y. Ans. $x = 10$; $y = 4$.

7. Given $3x + 5y = 40$, and $x + 2y = 14$, to find the values of x and y.

8. Given $5x + 3y = 0$, and $x - y = 8$, to find the values of x and y. Ans. $x = 3$; $y = -5$.

9. Given $6x + 5y = 77$, and $4x - 3y = 7$, to find the values of x and y. Ans. $x = 7$; $y = 7$.

10. Given $\frac{x}{3} + \frac{2y}{5} = 6$, and $\frac{2x}{3} + \frac{y}{5} = 6$, to find the values of x and y. Ans. $x = 6$; $y = 10$.

11. Given $\frac{x+2}{3} + 8y = 31$, and $\frac{y+5}{4} + 10x = 192$, to find the values of x and y. Ans. $x = 19$; $y = 3$.

12. Given $\frac{x}{3} + \frac{y}{5} - 5 = 0$, and $2x + \frac{y}{3} - 17 = 0$, to find the values of x and y. Ans. $x = 6$; $y = 15$.

13. Given $\frac{x+y}{8} - \frac{x-y}{2} = 0$, and $\frac{x+y}{4} + \frac{x-2y}{4} = 1\frac{3}{4}$, to find the values of x and y. Ans. $x = 5$; $y = 3$.

CASE III.

174. To eliminate by addition and subtraction.

1. Given $6x + 4y = 56$, and $4x - 3y = 9$, to find the values of x and y.

OPERATION.

$$\begin{array}{rcrl} 6x+4y &=& 56 & (1)\\ 4x-3y &=& 9 & (2)\\ \hline 12x+8y &=& 112 & (3)\\ 12x-9y &=& 27 & (4)\\ \hline 17y &=& 85 & (5)\\ y &=& 5 & (6)\\ 4x-15 &=& 9 & (7)\\ 4x &=& 24 & (8)\\ x &=& 6 & (9) \end{array}$$

Multiplying both members of equation (1) by 2, and of equation (2) by 3, we obtain equations (3) and (4), in which the coefficients of x are the same. Now, since the coefficients of x in these equations have like signs, we cancel the terms containing x, by subtracting (4) from (3), member from member (Art. 151), and obtain (5), an equation with only one unknown quantity, y. Reducing, we have (6), or $y=5$. Substituting 5 for y in equation (2), and reducing, we have (9), or $x=6$.

2. Given $6x+4y=32$, and $4x-2y=12$, to find x and y.

OPERATION.

$$\begin{array}{rcrl} 6x+4y &=& 32 & (1)\\ 4x-2y &=& 12 & (2)\\ 3x+2y &=& 16 & (3)\\ \hline 7x &=& 28 & (4)\\ x &=& 4 & (5)\\ 12+2y &=& 16 & (6)\\ 2y &=& 4 & (7)\\ y &=& 2 & (8) \end{array}$$

Dividing equation (1) by 2, we obtain (3), in which the coefficient of y has been made the same as in (2). Since the coefficients of y in these equations have different signs, we can cancel the terms containing y, by adding (2) and (3) together, member to member (Art. 151), and thus obtain (4), an equation with only one unknown quantity, x. Reducing, we have (5), or $x=4$. Substituting 4 for x in equation (3), and reducing, we have (8), or $y=2$. Hence the

RULE.

Multiply or divide one or both of the equations, if necessary, by such a number or quantity that one of the unknown quanti-

---

Explain the operation of Example 1. Of Example 2. What is the Rule?

ties shall have the same coefficient in both. Then, if the signs of the terms having the same coefficients are alike, subtract one equation from the other; or, if unlike, add the two equations together.

Note. If the coefficients of the quantity to be eliminated are prime to each other, each equation must be multiplied by the coefficient found in the other equation. In general, such a multiplier may be used for each as will produce the least common multiple of the coefficients.

If we wish to avoid fractions, it is convenient to divide only when one of the equations is not reduced to its simplest form, that is, when all its terms are exactly divisible by some quantity, as in Examples 3 and 4, below.

Examples.

3. Given $4x + 3y = 25$, and $12x - 6y = 30$, to find the values of x and y. Ans. $x = 4$; $y = 3$.

4. Given $3x - y = 22$, and $2x + 4y = 24$, to find the values of x and y. Ans. $x = 8$; $y = 2$.

5. Given $x + 8y = 44$, and $6x + y = 29$, to find the values of x and y. Ans. $x = 4$; $y = 5$.

6. Given $23x - 8y = 70$, and $8x - 2y = 40$, to find the values of x and y. Ans. $x = 10$; $y = 20$.

7. Given $4x - 5y = 0$, and $x - y = 1$, to solve the equations.

8. Given $x + y = 35$, and $\frac{3x}{7} + \frac{9y}{14} = 18$, to solve the equations. Ans. $x = 21$; $y = 14$.

9. Given $\frac{7x}{8} + \frac{4y}{9} = 29$, and $\frac{11x}{12} - \frac{5y}{6} = 7$, to find x and y. Ans. $x = 24$; $y = 18$.

175. Find the values of the unknown quantities in each of the following equations, by any of the methods of elimination.

What is the Note?

1. Given $\left\{\begin{array}{l} 2x + y = 35 \\ 5x - 3y = 27 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 12. \\ y = 11. \end{array}\right.$

2. Given $\left\{\begin{array}{l} 5y - 5x = 15 \\ 3x + 5y = 71 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 7. \\ y = 10. \end{array}\right.$

3. Given $\left\{\begin{array}{l} \frac{x}{6} + \frac{y}{4} = 6 \\ \frac{x}{4} + \frac{y}{6} = 5\frac{2}{3} \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 12. \\ y = 16. \end{array}\right.$

4. Given $\left\{\begin{array}{r} 3x + 2y = 4 \\ 4y - 3x + 1 = 0 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 1. \\ y = \frac{1}{2}. \end{array}\right.$

5. Given $\left\{\begin{array}{r} -4x + 3y = 45 \\ 2y + 6x = 4 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = -3. \\ y = 11. \end{array}\right.$

6. Given $\left\{\begin{array}{l} \frac{x}{2} + y = 35 \\ \frac{y}{3} + x = 45 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 40. \\ y = 15. \end{array}\right.$

7. Given $\left\{\begin{array}{l} 10x = 2 + 2y \\ 4y = 20 - 4x \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 1. \\ y = 4. \end{array}\right.$

8. Given $\left\{\begin{array}{l} x + y = a \\ x - y = b \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = \frac{a+b}{2}. \\ y = \frac{a-b}{2}. \end{array}\right.$

9. Given $\left\{\begin{array}{l} 7x - 3y = 26 \\ 2x - 2y = 6\frac{2}{3} \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 4. \\ y = \frac{2}{3}. \end{array}\right.$

10. Given $\left\{\begin{array}{l} \frac{3x}{4} - \frac{y}{2} = 9 \\ 2x - 2y = 16 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 20. \\ y = 12. \end{array}\right.$

11. Given $\left\{\begin{array}{l} \frac{x}{4} + \frac{y}{5} = 5 \\ \frac{2x}{3} + y = 18 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 12. \\ y = 10. \end{array}\right.$

12. Given $\left\{\begin{array}{l} 2y + 79 = 5x \\ 3x - 7 = 4 + x + y \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 57. \\ y = 103. \end{array}\right.$

13. Given $\left\{\begin{aligned} \frac{1}{x}+\frac{2}{y}&=\frac{11}{15} \\ \frac{3}{x}+\frac{4}{y}&=\frac{9}{5} \end{aligned}\right\}.$ Ans. $\left\{\begin{aligned} x&=3. \\ y&=5. \end{aligned}\right.$

Note. Eliminate before clearing of fractions.

14. Given $\left\{\begin{aligned} 2x^{-1}+3y^{-1}&=\tfrac{3}{2} \\ 4x^{-1}+8y^{-1}&=\tfrac{10}{3} \end{aligned}\right\}.$ Ans. $\left\{\begin{aligned} x&=2. \\ y&=6. \end{aligned}\right.$

15. Given $\left\{\begin{aligned} \frac{x}{2}+\frac{y}{3}&=a \\ \frac{x}{3}+\frac{y}{4}&=b \end{aligned}\right\}.$ Ans. $\left\{\begin{aligned} x&=18a-24b. \\ y&=36b-24a. \end{aligned}\right.$

16. Given $\left\{\begin{aligned} \frac{2x}{5}+\frac{3y}{4}&=\frac{9}{20} \\ \frac{3x}{4}+\frac{2y}{5}&=\frac{61}{120} \end{aligned}\right\}.$ Ans. $\left\{\begin{aligned} x&=\tfrac{1}{2}. \\ y&=\tfrac{1}{3}. \end{aligned}\right.$

17. Given $\left\{\begin{aligned} \frac{4x+y}{10}+x-5y&=33\tfrac{1}{4} \\ \frac{8x+12y}{15}+9y&=1 \end{aligned}\right\}.$ Ans. $\left\{\begin{aligned} x&=20\tfrac{1}{4}. \\ y&=-1. \end{aligned}\right.$

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

176. In a problem expressing two conditions, two required quantities may be so related, that, one of them being found, the other may be readily derived from it; in which case the solution can be effected by means of a single letter.

There are, however, certain problems whose solution requires each of the unknown quantities to be represented by its own proper symbol, and the formation of as many independent equations as there are unknown quantities.

1. A merchant sold at one time 3 hats and 4 caps for $ 23, and at another time 2 hats and 7 caps for $ 24; what was the price of each?

SOLUTION.

Let $x =$ the price of a hat,
and $y =$ the price of a cap.

Then, $3x + 4y = 23$ (1)
and $2x + 7y = 24$ (2)

Transposing and dividing (1), $x = \frac{23 - 4y}{3}$ (3)

Transposing and dividing (2), $x = \frac{24 - 7y}{2}$ (4)

Equating, $\frac{23 - 4y}{3} = \frac{24 - 7y}{2}$ (5)

Clearing of fractions, $46 - 8y = 72 - 21y$ (6)

Reducing, $13y = 26$ (7)

Whence, $y = 2$ (8)

Substituting 2 for y in (4), $x = \frac{24 - 14}{2}$ (9)

Whence, $x = 5$ (10)

2. The sum of two numbers is 133, and their difference is 47; required the numbers. Ans. 90 and 43.

3. A farmer paid 4 men and 6 boys 72 shillings for laboring one day, and afterwards, at the same rate, he paid 3 men and 9 boys 81 shillings for one day; what were the wages of each?

Ans. Men's wages, 9 shillings; boys', 6 shillings.

4. The value of my two horses is such that, if the value of the first be added to four times the value of the second, the sum is $ 580; and if the value of the second be added to four times that of the first, the sum is $ 520; required the value of each.

Ans. The first, $ 100; the second, $ 120.

Explain the solution of Problem 1.

5. Find that number, consisting of two figures, to which, if the number formed by changing the place of the figures be added, the sum is 121; and if it is subtracted, the remainder is 9.

SOLUTION.

Let $x =$ the first figure,
and $y =$ the second figure.
Then $10x =$ the first in tens' place,
and $10y =$ the second in tens' place.
Therefore $10x + y =$ the number required,
and $10y + x =$ the number formed.
Hence $11x + 11y =$ the sum of the numbers,
and $9x - 9y =$ the difference of the numbers.

Then, $11x + 11y = 121$ (1)
and $9x - 9y = 9$ (2)
Dividing (1), $x + y = 11$ (3)
Dividing (2), $x - y = 1$ (4)
Adding (3) and (4), $2x = 12$ (5)
Whence, $x = 6$ (6)
Subtracting (4) from (3), $2y = 10$ (7)
Whence, $y = 5$ (8)
Therefore, $10x + y = 65$ (9)

6. There is a number consisting of two figures, which is equal to four times the sum of those figures; and if 9 be subtracted from twice the number, the places of the figures will be reversed; what is the number?

Ans. 36.

7. A gentleman asked a lady her age; she replied: "7 years ago I was three times as old as you, but if we live 7 years longer, my age will be twice as great as yours"; what were their ages?

Ans. Lady's age, 49 years; gentleman's, 21 years.

Explain the solution of Problem 5.

8. A said to B, "If $\frac{1}{4}$ of your money were added to $\frac{1}{6}$ of mine, the sum would be \$6." B replied, "If $\frac{1}{4}$ of yours were added to $\frac{1}{6}$ of mine, the sum would be \$$5\frac{2}{3}$." What sum had each? Ans. A, \$12; B, \$16.

9. I have in two purses \$84; and if the sum in the purse containing the most be divided by the sum in the other, the quotient will be 13. Required the sum in each purse? Ans. In one, \$78; in the other, \$6.

10. The ages of a father and his son added together equal 140 years; and the age of the father is to that of the son as 3 to 2.

SOLUTION.

Let $x =$ the age of the father,
and $y =$ the age of the son.

Then, $x + y = 140$ (1)
and $x : y = 3 : 2$ (2)
Or, $3y = 2x$ (3)

Dividing (3), $y = \frac{2x}{3}$ (4)
Substituting $\frac{2x}{3}$ for y in (1), $x + \frac{2x}{3} = 140$ (5)
Reducing (5), $5x = 420$ (6)
Whence, $x = 84$ (7)
From (4), $y = 56$ (8)

11. The age of James is to that of John as 3 to 4; but 6 years hence their ages will be in the ratio of 5 to 6. What are their ages?

Ans. James's age, 9 years; John's 12 years.

12. Find two numbers, the greater of which shall be to 24 as their sum to 42, and the difference of which shall be to 6 as 4 to 3. Ans. 32 and 24.

Explain the solution of Problem 10.

13. If 3 be added to the numerator of a certain fraction, its value will be $\frac{1}{3}$; and if 1 be subtracted from the denominator, its value will be $\frac{1}{5}$. What is the fraction?

SOLUTION.

Let $x =$ the numerator,
and $y =$ the denominator.
Therefore $\frac{x}{y} =$ the fraction.

Then, $$\frac{x+3}{y} = \frac{1}{3} \qquad (1)$$

and $$\frac{x}{y-1} = \frac{1}{5} \qquad (2)$$

Clearing (1) of fractions, $3x + 9 = y$ (3)
Clearing (2) of fractions, $5x = y - 1$ (4)
Subtracting (3) from (4), $2x - 9 = -1$ (5)
Or, $2x = 8$ (6)
Whence, $x = 4$ (7)
From (3), $y = 21$ (8)
Hence, $\frac{x}{y} = \frac{4}{21}$ (9)

14. Divide 72 into two such parts that 3 times the greater shall exceed twice the less by 121.

Ans. 53 and 19.

15. Fifty laborers were engaged to remove an obstruction on a railroad; some of them by agreement were to receive \$0.90, and others, \$1.50. There was paid them just \$48, but no memorandum having been made, it is required to find how many worked at each rate.

Ans. For \$0.90, 45; for \$1.50, 5.

16. The wages of 5 men and 7 women amount to \$16.40, and 7 men receive more than 6 women by \$4. What does each receive?

Ans. Men, \$1.60; women, \$1.20.

Explain the solution of Problem 13.

C

17. If 4 be added to the numerator of a certain fraction, its value will be $\frac{1}{2}$; and if 7 be added to its denominator, its value will be $\frac{1}{5}$. What is that fraction?

Ans. $\frac{5}{18}$.

18. A sum of money was divided equally among a certain number of persons; had there been three more, each would have received \$1 less, and had there been two fewer, each would have received \$1 more than he did; required the number of persons, and what each received.

SOLUTION.

Let $x =$ number of persons,

and $y =$ no. dollars each received;

also, $xy =$ sum divided.

Then, $(x+3)(y-1) = xy$ (1)

and $(x-2)(y+1) = xy$ (2)

From (1), $xy + 3y - x - 3 = xy$ (3)

From (2), $xy - 2y + x - 2 = xy$ (4)

Transposing in (3), $3y - x = 3$ (5)

Transposing in (4), $x - 2y = 2$ (6)

Adding (5) and (6), $y = 5$ (7)

From (6), $x = 12$ (8)

19. My income tax and assessed tax together amount to \$30; but if the income tax were increased 20 per cent, and the assessed tax were decreased 25 per cent, the two together would amount to \$$32\frac{1}{8}$; required the amount of each tax.

Ans. Income tax, \$$21\frac{7}{18}$; assessed tax, \$$8\frac{11}{18}$.

20. Required two quantities such that, if the first be increased by *a*, it will become *m* times the second; and if the second be increased by *b*, it will become *n* times the first.

Ans. $\frac{a+bm}{mn-1}$ and $\frac{b+an}{mn-1}$.

Explain the Solution of Problem 18.

21. A has $\frac{4}{9}$ as much money as B; but if A should gain \$10 and B lose the same sum, they will have equal amounts. How much has each?

Ans. A, \$16; B, \$36.

22. A man and his wife can consume certain provisions in 15 days; but after partaking of them for 6 days, the woman consumed the remainder in 30 days. In what time could either consume the whole?

Ans. The man, in $21\frac{3}{7}$ days; his wife, in 50 days.

Eliminate before clearing of fractions, if the unknown quantities appear as denominators in each equation. Or, use negative exponents. (See Ex. 13, 14, Art. 175).

23. A merchant has sugar at a cents a pound and at b cents a pound; how much of each must he take to make a mixture of d pounds, worth c cents a pound?

Ans. At a cents, $\frac{d(c-b)}{a-b}$; at b cents, $\frac{d(a-c)}{a-b}$.

24. A composition of copper and tin, containing 100 cubic inches, weighed 505 ounces; how many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ oz., and a cubic inch of tin to weigh $4\frac{1}{4}$ oz.?

Ans. Copper, 420 oz.; tin, 85 oz.

25. There is a rectangular garden of a certain size; if it were 5 feet broader and 4 feet longer, it would contain 116 square feet more; and if it were 4 feet broader and 5 feet longer, it would contain 113 square feet more. Required its dimensions.

Ans. Length, 12 feet; breadth, 9 feet.

26. A person possesses certain capital which is invested at a certain rate per cent. A second person has \$1000 more capital than the first person, and invests it at *one per cent more;* thus his income exceeds that of the first person by \$80. A third person has \$1500 more capital than the first, and invests it at *two per cent more;* thus

his income exceeds that of the first person by \$150. Required the sum of each person, and the rate at which it is invested.

Ans. $\begin{cases} \text{Sums, \$3000, \$4000, and \$4500.} \\ \text{Rates, 4, 5, and 6 per cent.} \end{cases}$

T

SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

177. Any of the methods which have been given for the solution of simple equations containing two unknown quantities may be extended to those containing three or more unknown quantities.

1. Given $\left\{\begin{array}{r} x+y+z=6 \\ x+2y+3z=14 \\ 3x-y+4z=13 \end{array}\right\}$ to find x, y, and z.

OPERATION.

$$\begin{aligned}
x+y+z&=6 &&(1)\\
x+2y+3z&=14 &&(2)\\
3x-y+4z&=13 &&(3)\\
3x+3y+3z&=18 &&(4)\\
y+2z&=8 &&(5)\\
-4y+z&=-5 &&(6)\\
-8y+2z&=-10 &&(7)\\
9y&=18 &&(8)\\
y&=2 &&(9)\\
2+2z&=8 &&(10)\\
2z&=6 &&(11)\\
z&=3 &&(12)\\
x+2+3&=6 &&(13)\\
x&=1 &&(14)
\end{aligned}$$

By multiplying equation (1) by 3, we obtain equation (4), and by subtracting (1) from (2), and (4) from (3), we have (5) and (6), equations containing only two unknown quantities. Multiplying (6) by 2, and subtracting the product (7) from (5), give (8), an equation containing only one unknown quantity, y. Dividing (8) by 9, we have (9), or $y=2$. Substituting 2 for y in (5) gives (10), and reducing, we have (12), or $z=3$. Substituting 2 for y, and 3 for z in (1), and reducing, we have (14), or $x=1$.

Explain the operation.

2. Given $\left\{\begin{array}{l} x + y + z = 53 \\ x + 2y + 3z = 107 \\ x + 3y + 4z = 137 \end{array}\right\}$ to find the values of x, y, and z.

OPERATION.

$$
\begin{array}{rclr}
x + y + z & = & 53 & (1) \\
x + 2y + 3z & = & 107 & (2) \\
x + 3y + 4z & = & 137 & (3) \\
\hline
x & = & 53 - y - z & (4) \\
x & = & 107 - 2y - 3z & (5) \\
x & = & 137 - 3y - 4z & (6) \\
\hline
53 - y - z & = & 107 - 2y - 3z & (7) \\
107 - 2y - 3z & = & 137 - 3y - 4z & (8) \\
y & = & 54 - 2z & (9) \\
y & = & 30 - z & (10) \\
30 - z & = & 54 - 2z & (11) \\
z & = & 24 & (12) \\
y & = & 6 & (13) \\
x & = & 23 & (14)
\end{array}
$$

By transposing terms in (1), (2), and (3), we obtain equations (4), (5), and (6). Equating the second members of (4) and (5), and those of (5) and (6), we have (7) and (8), equations containing only two unknown quantities. Transposing terms in (7) and (8), we have (9) and (10). Equating the second members of (9) and (10) gives (11), an equation with only one unknown quantity, which reduces to (12), or $z = 24$. Substituting 24 for z in (10), and reducing, we have (13), or $y = 6$; and substituting 24 for z, and 6 for y, in (4), and reducing, we have (14), or $x = 23$.

From the preceding examples and illustrations, we deduce the following

RULE.

Deduce from the given equations, by elimination, a new set of equations containing one less unknown quantity; and continue

---

Explain the operation. Repeat the Rule.

the process until an equation is obtained containing but one unknown quantity.

Find the value of the unknown quantity in this equation. By substituting this value in either one of the set of two equations containing two unknown quantities, find the value of a second unknown quantity. Then, by substituting these values in either of the equations which contain three unknown quantities, find the value of a third; and so on, till the values of all are found.

NOTE. Upon the good judgment and discrimination of the learner in selecting the quantity to be first eliminated, and the method of elimination suited to the particular case, will depend the simplicity and elegance of the solution.

EXAMPLES.

Find the values of the unknown quantities in the following equations.

3. Given $\left\{\begin{array}{l} x + 2y + z = 24 \\ 2x + y + 3z = 38 \\ 3x + 3y + 2z = 46 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 4. \\ y = 6. \\ z = 8. \end{array}\right.$

4. Given $\left\{\begin{array}{l} 4x + 2y - z = 26 \\ 5x + 2y - 3z = 16 \\ 2x - y + 2z = 23 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 6. \\ y = 5. \\ z = 8. \end{array}\right.$

5. Given $\left\{\begin{array}{l} x + y + z = 33 \\ y - x + z = 23 \\ z - x - y = 1 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 5. \\ y = 11. \\ z = 17. \end{array}\right.$

6. Given $\left\{\begin{array}{l} u + x + y = 6 \\ u + x + z = 9 \\ u + y + z = 8 \\ x + y + z = 7 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} u = 3. \\ x = 2. \\ y = 1. \\ z = 4. \end{array}\right.$

The solution may here be abridged, by the artifice of assuming the sum of the four unknown quantities to equal s.

What is the Note? Explain the operation of Example 6.

Thus, $u + x + y + z = s$

Then the first equation is $s - z = 6$ (1)

The second is $s - y = 9$ (2)

The third is $s - x = 8$ (3)

The fourth is $s - u = 7$ (4)

By addition, $4s - s = 30$ (5)

Whence, $s = 10$ (6)

Substituting the value of s in (4), (3), (2), and (1), and reducing, we have $u = 3$, $x = 2$, $y = 1$, and $z = 4$.

7. Given $\left[\begin{array}{l} x + y + z = 13 \\ u + x + y = 17 \\ u + x + z = 18 \\ u + y + z = 21 \end{array}\right]$. Ans. $\left\{\begin{array}{l} x = 2. \\ y = 5. \\ z = 6. \\ u = 10. \end{array}\right.$

8. Given $\left\{\begin{array}{r} 7x - 3y - z = 12 \\ x + 2y + 3z = 17 \\ 4x - y + 2z = 13 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 4. \\ y = 5. \\ z = 1. \end{array}\right.$

9. Given $\left\{\begin{array}{l} x + y - z = 0 \\ x + z - y = 2 \\ y + z - x = 4 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 1. \\ y = 2. \\ z = 3. \end{array}\right.$

10. Given $\left[\begin{array}{l} \frac{x}{a} + \frac{y}{b} = 1 \\ \frac{x}{a} + \frac{z}{c} = 1 \\ \frac{y}{b} + \frac{z}{c} = 1 \end{array}\right]$. Ans. $\left[\begin{array}{l} x = \frac{a}{2}. \\ y = \frac{b}{2}. \\ z = \frac{c}{2}. \end{array}\right.$

Note. Eliminate before clearing of fractions.

11. Given $\left\{\begin{array}{l} x + y = a \\ x + z = b \\ y + z = c \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = \frac{1}{2}(a + b - c). \\ y = \frac{1}{2}(a + c - b). \\ z = \frac{1}{2}(b + c - a). \end{array}\right.$

12. Given $\left[\begin{array}{l} \frac{x}{4} + \frac{y}{3} + \frac{z}{2} = 9 \\ \frac{2x}{3} - \frac{y}{2} + \frac{3z}{4} = 11 \\ \frac{3x}{4} + \frac{2y}{3} - \frac{z}{2} = 9 \end{array}\right]$. Ans. $\left\{\begin{array}{l} x = 12. \\ y = 6. \\ z = 8. \end{array}\right.$

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

178. Problems leading to simple equations containing three or more unknown quantities require precisely analogous processes in their solution to those required by problems leading to simple equations containing two unknown quantities.

1. Three boys, James, Henry, and Arthur, bought fruit at the same prices. James paid for 3 oranges, 1 apple, and 2 pears, 14 cents; Henry paid for 4 oranges, 3 apples, and 1 pear, 17 cents; and Arthur paid for 1 orange, 4 apples, and 3 pears, 13 cents. What was the price of each?

Ans. Oranges, 3 cents; apples, 1 cent; pears, 2 cents.

2. A gentleman divided $100 among his four daughters, Mary, Isabel, Jane, and Ellen, in such a manner, that twice Isabel's part added to three times Ellen's part was $160; three times Mary's part added to twice Jane's part was $90; twice Mary's part added to Ellen's part was $60. What sum did each receive?

Ans. Mary, $10; Isabel, $20; Jane, $30; Ellen, $40.

3. I have three ingots, composed of different metals. A pound of the first contains 7 ounces of silver, 3 ounces of copper, and 6 ounces of tin; a pound of the second contains 12 ounces of silver, 3 ounces of copper, and 1 ounce of tin; and a pound of the third contains 4 ounces of silver, 7 ounces of copper, and 5 ounces of tin. How much of each of these three ingots must be taken in order to form a fourth, each pound of which shall contain 8 ounces of silver, $3\frac{3}{4}$ ounces of copper, and $4\frac{1}{4}$ ounces of tin?

Ans. Of the first, 8 ounces; of the second, 5 ounces; and of the third, 3 ounces.

Let x, y, and z denote the number of ounces that must be taken of each of the three ingots, respectively. Then, since there are 7 ounces of silver in a pound, or 16 ounces, of the first ingot, in 1 ounce of it there are $\frac{7}{16}$ of an ounce of silver, and, consequently, in x ounces there are $\frac{7x}{16}$ of an ounce of silver. In like manner, we may find that $\frac{12y}{16}$ and $\frac{4z}{16}$ denote the number of ounces of silver to be taken of the second and third; but, by the problem, one pound of the fourth ingot is to contain 8 ounces of silver; hence we have for the first equation,

$$\frac{7x}{16}+\frac{12y}{16}+\frac{4z}{16}=8.$$

Proceeding in like manner with respect to the copper and tin, we have for the other equations,

$$\frac{3x}{16}+\frac{3y}{16}+\frac{7z}{16}=\frac{15}{4},$$

$$\frac{6x}{16}+\frac{y}{16}+\frac{5z}{16}=\frac{17}{4}.$$

From these equations, the results given above are readily obtained.

4. A gentleman purchased a chaise, horse, and harness for $400. He paid four times as much for the chaise as for the harness, and one third as much for the harness as for the horse. How much did he pay for each?

Ans. Chaise, $200; horse, $150; harness, $50.

5. There are three numbers whose sum is 324; the second exceeds the first as much as the third exceeds the second; and the first is to the third as 5 to 7. What are the numbers? Ans. 90, 108, and 126.

6. A man speaking with his wife and son respecting their ages, said that his age added to that of his son was 12 years more than that of his wife; the wife said that her age added to that of her son was 8 years more than that of her husband, and that their ages together amounted to 92 years. Required the age of each.

Ans. Husband, 42 years; wife, 40 years; son, 10 years.

7. A bin holding 146 bushels is filled with a mixture of wheat, barley, and oats. The barley exceeds the wheat by 15 bushels, and there are as many bushels of oats as of both wheat and barley. What is the quantity of each?

Ans. Wheat, 29 bushels; barley, 44 bushels; and oats, 73 bushels.

8. A and B can perform a piece of work in 8 days, A and C in 9 days, and B and C in 10 days; in how many days can each alone perform it?

Ans. A, in $14\frac{34}{49}$ days; B, in $17\frac{23}{41}$ days; and C, in $23\frac{7}{31}$ days.

9. A certain number consists of three digits, whose sum is 9. If 198 be subtracted from the number, the remainder will consist of the same digits in a reverse order; and if the number be divided by the digit at the left, the quotient is 108. What is the number?

Ans. 432.

Let x, y, and z denote the digits, respectively, beginning at the left; then, $100x + 10y + z =$ the number.

10. I have three horses, and a carriage, which of itself is worth \$220. If I put the carriage with the first horse, it will make the value equal to that of the second and third; but if I put it with the second horse, it will make the value double that of the first and third; and if I put it with the third horse, it will make the value triple that of the first and second. What is the value of each horse?

Ans. First, \$20; second, \$100; third, \$140.

11. A and B can reap a certain field in a days, A and C in b days, and B and C in c days; in what time can each alone reap it?

Ans. A, in $\frac{2abc}{ac+bc-ab}$ days; B, in $\frac{2abc}{ab+bc-ac}$ days; C, in $\frac{2abc}{ab+ac-bc}$ days.

12. Find three numbers such that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third shall be equal to 62; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third shall be equal to 47; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third shall be equal to 38. Ans. 24, 60, and 120.

13. Three boys, A, B, and C, owe, together, $2.19, and no one of them has so much money. But by uniting, it is found that it can be paid in several ways; first, by $\frac{3}{7}$ of B's money and all of A's; secondly, by $\frac{5}{9}$ of C's money and all of B's; or, thirdly, by $\frac{2}{3}$ of A's money and all of C's. How much money has each?

Ans. A, $1.53; B, $1.54; and C, $1.17.

14. Find four numbers, such that the first, together with half the second, may be equal to 357; the second, with $\frac{1}{3}$ of the third, equal to 476; the third, with $\frac{1}{4}$ of the fourth, equal to 595; and the fourth, with $\frac{1}{5}$ of the first, equal to 714.

Ans. First number, 190; second, 334; third, 426; fourth, 676.

15. A merchant has three kinds of sugar. He can sell 3 lbs. of the first quality, 4 lbs. of the second quality, and 2 lbs. of the third quality, for 60 cents; or, he can sell 4 lbs. of the first quality, 1 lb. of the second quality, and 5 lbs. of the third quality, for 59 cents; or, he can sell 1 lb. of the first quality, 10 lbs. of the second quality, and 3 lbs. of the third quality, for 90 cents. Required the price of each quality.

Ans. First quality, 8 cts. per lb.; second, 7 cts.; third, 4 cts.

16. A, B, and C engaged in a squirrel hunt, and killed 96 squirrels, which they wish to share equally. In order to do this, A, who has most, gives to B and C as many as they each already had; then, B gives to A and C as many as they each had after the first division; and, lastly, C gives to A and B as many as each had after the second division, when it was found that each had the same number. How many had each?

Ans. A, 52; B, 28; and C, 16.

DISCUSSION

OF SOME PROBLEMS LEADING TO SIMPLE EQUATIONS.

179. The Discussion of a problem consists in attributing various values and relations to the known quantities entering into the general equation, and in interpreting the results.

INTERPRETATION OF NEGATIVE RESULTS.

180. The interpretation of negative results obtained by means of *simple equations* is illustrated in the problems which follow.

1. Let it be required to find what number must be added to the number a, that the sum may be b.

Let $x =$ the required number.

Then, $a + x = b$,

whence, $x = b - a$.

Here, the value of x corresponds to any assigned values of a and b. Thus, for example,

Let $a = 12$ and $b = 25$.

Then, $x = 25 - 12 = 13$,

which satisfies the conditions of the problem, for if 13 be added to 12, or a, the sum will be 25, or b.

But suppose $a = 30$ and $b = 24$.

Then $x = 24 - 30 = -6$,

which indicates that, under the latter hypothesis, the problem is impossible in an *arithmetical sense*, though it is possible in the *algebraic sense* of the words "number," "added," and "sum."

In what does the Discussion of a problem consist? Give the discussion of the first problem, stating what the negative result points out, and correcting the enunciation.

The negative result, -6, points out, therefore, either *an error* or *an impossibility.*

But, taking the value of x with a contrary sign, we see that it will satisfy the enunciation of the problem, in an arithmetical sense, when modified so as to read:

What number must be taken from 30, that *the difference* may be 24?

2. Let it be required to find the epoch at which A's age is twice as great as B's, A's age at present being 35 years, and B's 20 years.

Let us suppose the required epoch to be *after* the present date.

Then $x =$ the number of years *after* the present date,

and $35 + x = 2(20 + x)$;

whence, $x = -5$,

a negative result.

On recurring to the problem, we find it is so worded as to admit also of the supposition that the epoch is *before* the present date, and taking the value of x obtained, with the contrary sign, we find it will satisfy that enunciation.

Hence, a negative result here indicates that a wrong choice was made of two possible suppositions which the problem allowed.

From the foregoing examples and illustrations we may infer:—

1. *That negative results indicate either an erroneous enunciation of a problem, or a wrong supposition respecting the quality of some quantity belonging to it.*

2. *That we may form a possible problem analogous to that which involved the impossibility, or correct the wrong supposition, by attributing to the unknown quantity in the equation a quality directly opposite to that which had been attributed to it.*

3. *That the true answer of the corrected problem will be found by simply changing the sign of the negative result obtained.*

Give the discussion of the second problem, and show what the negative result indicates. What may be inferred from the examples and illustrations?

Interpret the negative answers obtained, and modify the enunciation so as to give positive results, for the following

PROBLEMS.

3. What number must be taken from 10, that the remainder shall be 15? Ans. -5.

4. What number is that whose fifth part exceeds its fourth part by 4? Ans. -80.

5. A man at the time of his marriage was 40 years old, and his wife 36 years; how many years must elapse before his age will be to hers as 6 to 5? Ans. -16 years.

6. The length of a certain field is 8 rods, and its breadth 5 rods; how much must be added to its length that its contents may be 30 square rods? Ans. -2 rods.

7. What number is that, the sum of the third and fifth parts of which, diminished by 7, is equal to the original number? Ans. -15.

8. If 2 be added to the numerator of a certain fraction, its value is $\frac{1}{4}$; but if 2 be added to its denominator, its value is $\frac{1}{2}$. What is the fraction? Ans. $\frac{-5}{-12}$.

9. A father has lived 45 years, and his son 15 years. Find in how many years the age of the son will be one fourth of the age of the father. Ans. -5.

10. A man worked 12 days, his son being with him 8 days, and received $22, besides the subsistence of himself and son while at work. At another time he worked 10 days, and had his son with him 4 days, and received $19. What were the daily wages of each?

Ans. The father's wages, $2; the son's, —25 cts.

That is, the father earned $2 a day, and was at the expense of $0.25 a day for his son's subsistence.

e

ZERO AND INFINITY.

181. Zero, which is represented by the symbol 0, not only denotes absence of value, or nothing, but may, in Algebra, stand for a quantity less than any assignable value.

182. Infinity, which is represented by the symbol ∞, denotes a quantity greater than any assignable value.

In comparison with infinities, finite values may be considered as all equal to one another.

INTERPRETATION OF $\frac{A}{0}$, $\frac{A}{\infty}$, $\frac{0}{A}$, AND $\frac{0}{0}$.

183. In order to explain the meaning of these symbols, let us take the fraction $\frac{a}{b}$.

1. Suppose the numerator, a, to remain constant, while the denominator, b, continually decreases. Then, since the value of a fraction depends upon the relative value of its terms (Art 113), the fraction must increase as the denominator decreases; consequently, when b decreases below any determinate limits, the value of the fraction must exceed any determinate or assignable quantity. Hence, representing any finite quantity by A, we have

$$\frac{A}{0} = \infty.$$

That is,

If a finite quantity is divided by zero, the quotient is infinity.

2. Suppose the numerator, a, to remain constant, while the denominator, b, constantly increases. Then, the value of the fraction must decrease as the denominator increases; consequently, when b increases beyond any determinate limits, the value of the fraction must be less than any determinate or assignable quantity. Hence we have

$$\frac{A}{\infty} = 0.$$

Define Zero. Infinity. Interpret the symbol $\frac{A}{0}$. The symbol $\frac{A}{\infty}$.

That is,

If a finite quantity is divided by infinity, the quotient is zero.

3. Suppose, now, the denominator, b, to remain constant, while the numerator, a, constantly decreases. Then the value of the fraction must decrease as the numerator decreases; consequently, when a decreases below any determinate limits, the value of the fraction must be less than any determinate or assignable quantity. Hence we have

$$\frac{0}{A} = 0.$$

That is,

If zero is divided by a finite quantity, the quotient is zero.

4. Suppose, next, a and b both to decrease, at the same time and in the same ratio. Then, the value of the fraction will not be changed; but when a and b decrease below any determinate limits, the terms of the fractions each become zero, and the fraction itself becomes $\frac{0}{0}$. As $\frac{a}{b}$ may have any value, $\frac{0}{0}$ will represent any finite quantity. Hence,

If zero is divided by zero, the quotient may be any finite quantity.

NOTE. If $\frac{0}{0}$ is the result of an expression whose numerator contained more zero factors than its denominator, its value is 0; and if its denominator contained more zero factors than its numerator, its value is ∞. Sometimes $\frac{0}{0}$, by canceling a common factor in the terms of the fraction from which it originates, is found to have a definite, finite value.

184. From the foregoing discussion we draw the following inferences: —

1. *That a problem whose result appears under the form of* $\frac{A}{0}$ *is impossible, or cannot be satisfied by finite quantities.*

2. *That a problem whose result appears under the form of* $\frac{0}{0}$ *is generally indeterminate, or can be satisfied by any finite quantities whatever.*

Interpret the symbol $\frac{0}{A}$. The symbol $\frac{0}{0}$. What two inferences are drawn?

Interpret the results which may be obtained in the following

PROBLEMS.

1. Three railway companies issue a, b, and c shares, respectively, and the price paid on each is the same sum per share. But the second and third afterwards call for p dollars and q dollars per share, respectively, in addition to that originally paid, whereby the total capital paid up on the first and second together becomes double that paid up on the third. Required the price per share originally paid up.

Let $x =$ the price required.

Then, $ax + b(x + p) = 2c(x + q)$;

whence, $x = \dfrac{2cq - bp}{a + b - 2c}$.

If $a = 9000$, $b = 11000$, $c = 10000$, $p = 19$, and $q = 11$,

$$x = \frac{11000}{0} = \infty;$$

which shows that the problem, according to the conditions, is impossible.

Again, if $a = 9000$, $b = 11000$, $c = 10000$, $p = 10$, and $q = 5\frac{1}{2}$,

$$x = \frac{0}{0},$$

which shows that the conditions are satisfied without reference to the sum originally paid up; and that the unknown quantity may have any finite value whatever.

2. A is 60 years old, and B 40 years; when will they both be of the same age? Ans. ∞.

3. A person buys 400 sheep in two flocks; for the first he pays $1.50 per head, and for the second $2. Of the first he loses 30, and of the second 56. He then sells the remainder of the first flock at $2 per head, and of the second at $2.50 per head, and finds he has lost nothing. Required the number in each flock.

Ans. $\frac{0}{0}$.

INVOLUTION.

185. A Power of any quantity is the product obtained by taking that quantity one or more times as a factor. Thus,

$a = a^1$ is the first power of a,
$a\,a = a^2$ " second power, or square of a,
$a\,a\,a = a^3$ " third power, or cube of a,
$a\,a\,a\,a = a^4$ " fourth power of a;

and so on, the *exponent* (Art. 19) of the power denoting the number of times the quantity a is taken as a factor.

If the exponent is n, the power is the product of n factors, when n is any entire quantity whatever.

186. Involution is the process of raising a given quantity to any required power.

This may be effected, as is evident from the definition of a power, by taking the given quantity as a factor as many times as there are units in the exponent of the required power.

187. *When the quantity to be involved is positive, all the powers will be positive.*

For, any positive factor taken any number of times must always give a positive result (Art. 59). Thus,

$$(+a) \times (+a) = +a^2$$
$$(+a) \times (+a) \times (+a) = +a^3, \text{ and so on.}$$

188. *When the quantity to be involved is negative, all the even powers will be positive, and all the odd powers negative.*

For, a negative multiplier causes the sign of the product to be the opposite of that of the multiplicand (Art. 59), and therefore each

---

Define a Power. What does the exponent of the power denote? If the exponent is n, what is the power? Define Involution. When the quantity involved is positive, what sign do the powers take? When the quantity is negative?

additional negative factor changes the sign of the result. As the first power is negative, each odd power will be negative, and the even ones positive. Thus,

$$-a$$
$$(-a)\times(-a)=+a^2$$
$$(-a)\times(-a)\times(-a)=(+a^2)\times(-a)=-a^3$$
$$(-a)\times(-a)\times(-a)\times(-a)=(-a^3)\times(-a)=+a^4$$

and so on.

189. The involution of a polynomial, or of a monomial composed of several factors, is *indicated* by inclosing the quantity in a parenthesis, and writing the exponent at the right and a little above the expression. Thus,

$(a+b)^2$ indicates the second power of $a+b$;
$(2\,abc)^5$ indicates the fifth power of $2\,abc$.

POWERS OF MONOMIALS.

190. It is evident that the rules for involution must be based upon those for multiplication.

1. Let it be required to raise $3\,a^2b$ to the third power.

OPERATION.

$$\begin{aligned}(3\,a^2b)^3 &= 3\,a^2b\times 3\,a^2b\times 3\,a^2b\\ &= 3\times 3\times 3\,a^2a^2a^2bbb\\ &= 27\,a^6b^3.\end{aligned}$$

Since the required power is equivalent to the given quantity taken three times as a factor (Art. 185), we proceed, by the rule for multiplication (Art. 62), to find the product of $3\,a^2b\times 3\,a^2b\times 3\,a^2b$, or $27\,a^6b^3$; from which it appears,—

1. That the coefficient 3 has been raised to the third power.
2. That the exponent of each letter has been multiplied by 3, the exponent of the power.

How is the involution of a polynomial, or of a monomial composed of several factors, indicated? Explain the operation.

Hence, for raising a monomial to any power, we have the following

RULE.

Raise the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power.

NOTE. If the quantity involved is positive, all its powers will be positive (Art. 187); but if it is negative, all the *even* powers will be positive, and all the *odd* powers will be negative (Art. 188).

EXAMPLES.

2. Find the cube of $a\,b$. Ans. $a^3\,b^3$.
3. Find the square of $a\,x^2$. Ans. $a^2\,x^4$.
4. Find the fourth power of $x^2\,y$. Ans. $x^8\,y^4$.
5. Find the third power of $a\,b\,x^n$. Ans. $a^3\,b^3\,x^{3n}$.
6. Find the mth power of $c\,x^2\,y^n$. Ans. $c^m\,x^{2m}\,y^{mn}$.
7. Find the fifth power of $3\,a^2\,x^2$.
8. Raise $-3\,x$ to the third power. Ans. $-27\,x^3$.
9. Raise $-4\,x^2$ to the second power. Ans. $16\,x^4$.
10. Raise $a^2\,b^3\,c\,d^2$ to the fourth power. Ans. $a^8\,b^{12}\,c^4\,d^8$.
11. Required the cube of $-4\,a^2\,b^3\,x^4$. Ans. $-64\,a^6\,b^9\,x^{12}$.
12. Required the fourth power of $5\,a^2\,b^3\,c^4$. Ans. $625\,a^8\,b^{12}\,c^{16}$.
13. Required the square of $-3\,a\,b^2\,x$. Ans. $9\,a^2\,b^4\,x^2$.
14. Raise $2\,a\,b\,c^2$ to the sixth power. Ans. $64\,a^6\,b^6\,c^{12}$.

Repeat the Rule. The Note.

15. Raise $-2ax^2$ to the fourth power.

16. Required the fifth power of $4ax^2y$.

Ans. $1024a^5x^{10}y^5$.

17. Required the *n*th power of $-6a^3b^3$.

Ans. $\pm 6^na^{3n}b^{3n}$.

Note. Since *n* may be any number whatever (Art. 185), the *n*th power of the given negative quantity may be either even or odd, and therefore either positive or negative, as is indicated by the sign ±.

POWERS OF FRACTIONS.

191. Fractions, like entire quantities, are involved by multiplication.

1. Let it be required to find the third power of $\frac{2ax^2}{3bc}$.

OPERATION.

$$\left(\frac{2ax^2}{3bc}\right)^3 = \frac{2ax^2}{3bc} \times \frac{2ax^2}{3bc} \times \frac{2ax^2}{3bc}$$

$$= \frac{2ax^2 \times 2ax^2 \times 2ax^2}{3bc \times 3bc \times 3bc} = \frac{8a^3x^6}{27b^3c^3}.$$

Since the required power requires the given quantity to be taken three times as a factor (Art. 185), we find it by multiplying, as in multiplication of fractions (Art. 137). Hence the following

RULE.

Raise both the numerator and the denominator to the required power.

EXAMPLES.

2. Find the square of $\frac{3a^2b}{7d}$. Ans. $\frac{9a^4b^2}{49d^2}$.

Why does the answer to Example 17 have the sign ±? Explain the operation. Repeat the Rule.

3, Find the cube of $-\frac{3b^2}{c}$. Ans. $-\frac{27b^6}{c^3}$.

4. Find the square of $-\frac{2ax^2}{5b}$. Ans. $\frac{4a^2x^4}{25b^2}$.

5. Required the fifth power of $\frac{ac^n}{d^2}$. Ans. $\frac{a^5c^{5n}}{d^{10}}$.

6. Required the fourth power of $\frac{x^2y}{\frac{1}{2}b}$. Ans. $\frac{16x^8y^4}{b^4}$.

7. Required the third power of $\frac{-x}{2y}$. Ans. $\frac{-x^3}{8y^3}$.

8. Required the fifth power of $\frac{a^2cd^3}{2b^3}$.

9. Required the fourth power of $\frac{3}{4}a^3c^2$.

Ans. $\frac{81}{256}a^{12}c^8$.

10. Required the second power of $\frac{-6xy^2}{11cd}$.

Ans. $\frac{36x^2y^4}{121c^2d^2}$.

192. The rules already given hold true when any of the exponents are negative.

$$\text{For,}\quad (a^{-n})^m = \left(\frac{1}{a^n}\right)^m = \frac{1}{a^{mn}} = a^{-mn},$$

$$\text{and}\quad (a^n)^{-m} = \frac{1}{(a^n)^m} = \frac{1}{a^{mn}} = a^{-mn}. \quad \text{(Art. 71.)}$$

1. Required the third power of $5a^{-2}b^{-1}$.

Ans. $125a^{-6}b^{-3}$.

2. Required the fourth power of $-2c^3d^{-2}$.

Ans. $16c^{12}d^{-8}$.

3. Required the nth power of $-6ax^{-2}y^3$.

Ans. $\pm 6^n a^n x^{-2n} y^{3n}$.

4. Develop the expression $(-x^{-2n}y^n z)^7$.

Ans. $-x^{-14n}y^{7n}z^7$.

5. Develop the expression $(x^4y^{-3}z^{-1})^{-2}$.

Ans. $x^{-8}y^6z^2$.

6. Develop the expression $(-m^{-1}n^{-2})^{-3}$.

Ans. $-m^{3}n^{6}$.

7. Develop the expression $(-2a^{-3}b^{-2})^{-4}$.

Ans. $\frac{1}{16}a^{12}b^{8}$.

8. Required the sixth power of $\frac{a^{-n}c^{n}}{-b^{m}}$.

Ans. $\frac{a^{-6n}c^{6n}}{b^{6m}}$.

9. Raise $-\frac{5a^{-2}c^{2}d^{3}}{4x^{-2n}y^{2}z^{n}}$ to the third power.

Ans. $-\frac{125a^{-6}c^{6}d^{9}}{64x^{-6n}y^{6}z^{3n}}$.

10. What is the *m*th power of $\frac{2a^{-n}b^{n}c^{4}}{3x^{2n}y^{-2}z^{n-2}}$?

Ans. $\frac{2^{m}a^{-mn}b^{mn}c^{4m}}{3^{m}x^{2mn}y^{-2m}z^{mn-2m}}$.

POWERS OF BINOMIALS.

193. Binomials, like monomials, may be raised to any power by the process of successive multiplications.

Thus, $a+b$ raised to the second power is

$$(a+b)(a+b)=a^{2}+2ab+b^{2}.$$

And $a-b$ raised to the third power is

$$(a-b)(a-b)(a-b)=a^{3}-3a^{2}b+3ab^{2}-b^{3}.$$

But this process of involving binomials by actual multiplication must be very tedious, when high powers are required. There is, however, a much abridged process, discovered by Sir Isaac Newton, called

THE BINOMIAL THEOREM.

194. The Binomial Theorem expresses a general method of developing any power of a binomial.

How may binomials be raised to any power? What does the Binomial Theorem express?

195. With a view of elucidating the principles governing the development of Newton's theorem, we shall, by actual multiplication, find a few of the powers of a binomial, when both terms are positive; and also when one term is positive and the other negative.

1. Let $a+b$ be raised to the fifth power.

$$\begin{array}{lllllll}
a+b & & & & & \ldots\ldots\ldots & \text{1st power.}\\
a+b & & & & & & \\ \hline
a^2+ & ab & & & & & \\
\quad + & ab & +b^2 & & & & \\ \hline
a^2+ & 2ab & +b^2 & & & \ldots\ldots & \text{2d power.}\\
a+b & & & & & & \\ \hline
a^3+ & 2a^2b+ & ab^2 & & & & \\
\quad + & a^2b+ & 2ab^2 & +b^3 & & & \\ \hline
a^3+ & 3a^2b+ & 3ab^2 & +b^3 & & \ldots & \text{3d power.}\\
a+b & & & & & & \\ \hline
a^4+ & 3a^3b+ & 3a^2b^2+ & ab^3 & & & \\
\quad + & a^3b+ & 3a^2b^2+ & 3ab^3 & +b^4 & & \\ \hline
a^4+ & 4a^3b+ & 6a^2b^2+ & 4ab^3 & +b^4 & \ldots & \text{4th power.}\\
a+b & & & & & & \\ \hline
a^5+ & 4a^4b+ & 6a^3b^2+ & 4a^2b^3+ & ab^4 & & \\
\quad + & a^4b+ & 4a^3b^2+ & 6a^2b^3+ & 4ab^4 & +b^5 & \\ \hline
a^5+ & 5a^4b+ & 10a^3b^2+ & 10a^2b^3+ & 5ab^4 & +b^5 & \text{5th power.}
\end{array}$$

NOTE. In any binomial, as $a+b$, or $a-b$, the term at the left is called the *leading* letter or quantity, and the other the *following* letter or quantity.

2. Let $a-b$ be raised to the fifth power.

In what manner is $a+b$ raised to the fifth power? How is $a-b$ raised to the same power?

$a - b$
$a - b$

$a^2 - \quad ab$
$\quad - \quad ab +$

$a^2 - 2ab +$
$a - b$

$a^3 - 2a^2b +$
$\quad - \quad a^2b +$

$a^3 - 3a^2b +$
$a - b$

$a^4 - 3a^3b +$
$\quad - \quad a^3b +$

$a^4 - 4a^3b +$
$a - b$

$a^5 - 4a^4b + 6$
$\quad - \quad a^4b + 4$

$a^5 - 5a^4b + 10$

ity = quantity

unity

quantities

multiplication

exponent is a number

a letter

coefficient prefixed to it

In like manne
It will be seen
formly less by o
nent of the powe
that certain inva
other things:—

1. The numb
2. The signs
3. The letters
4. The exponents o
5. The coefficients of the terms.

What is seen with regard to the number of multiplications? What five other things follow invariable laws?

NUMBER OF THE TERMS.

196. By examining either of the examples, we observe that the *first* power has *two* terms ; the *second* power, *three* terms ; the *third* power, *four* terms ; the *fourth* power, *five* terms ; and the *fifth* power, *six* terms. Hence,

The number of terms is always one more than the exponent of the power.

SIGNS OF THE TERMS.

197. By examining the two examples, we observe that all the terms of the powers of $a + b$ are positive ; and of those of $a - b$, all the odd terms, reckoning from the left, are positive, and all the even terms are negative. Hence,

When both terms of the binomial are positive, all the terms of the power are positive.

When the second term of the binomial is negative, all the odd terms, reckoning from the left, are positive, and all the even terms negative.

LETTERS IN THE TERMS.

198. From the examination of the several powers, it is evident that

The leading letter or quantity enters all the terms of the power except the last; the following letter or quantity enters all the terms except the first; and the product of some powers of both letters compose all the intermediate terms.

EXPONENTS OF THE LETTERS.

199. By observing the different powers of $a + b$, and of $a - b$, we shall find that the exponents of the letters

---

What is the number of terms in any power of a binomial? What are the signs of the terms? In what manner do the letters enter into the terms? What is the law governing the exponents of the letters?

of the several terms follow an invariable order. Thus, in the fifth power of each of the binomials, the exponents are,

Of a, 5, 4, 3, 2, 1, 0;
Of b, 0, 1, 2, 3, 4, 5;

whose sum in each term is 5, or the same as the exponent of the power. Hence,

The exponent of the leading letter in the first term is the same as the exponent of the power, and decreases by one in each successive term to the right.

The exponent of the following letter in the second term is one, and increases by one in each successive term to the right, until the last, where the exponent is the same as that of the power.

The sum of the exponents in any term is the same as the exponent of the power.

COEFFICIENTS OF THE TERMS.

200. It will be observed that the coefficients of any power in the examples, as the fifth power, are,

Of the first term, a^5, 1;

Of the second term, $5a^4b$, the same as the exponent of the power, or 5;

Of the third term, $10a^3b^2$, the product of the coefficient of the preceding term by the exponent of the leading letter in that term, divided by 2, the number which marks the place of the term, or $\frac{5 \times 4}{2} = 10$; and, in like manner, the coefficient of any term. Hence,

The coefficient of the first term is one; that of the second term is the same as the exponent of the power; and, in general, the coefficient of any term is found by multiplying the

---

What is the law governing the coefficients of the terms?

coefficient of the preceding term by the exponent of the leading letter of the same term, and dividing the product by the number which marks its place.

Note 1. When the number of terms is even, there will be two terms in the middle, having the same coefficient; and since the same coefficients are repeated in an inverse order after passing the middle term or terms, most of the coefficients may be obtained without actual calculation.

Note 2. It will be seen that Theorems I. and II., Arts. 76 and 77, are only special cases coming under the Binomial Theorem.

Examples.

1. Raise $x-y$ to the third power.

OPERATION.

Coefficients and signs,	1	-3	$+3$	-1
x and its exponents,	x^3	x^2	x	
y and its exponents,		y	y^2	y^3
Combining,	$x^3-3x^2y+3xy^2-y^3$			

After a little practice, the learner can write out the final form at once.

2. Raise $x+y$ to the second power.

Ans. $x^2+2xy+y^2$.

3. Expand $(c-d)^4$.

Ans. $c^4-4c^3d+6c^2d^2-4cd^3+d^4$.

4. Required the fourth power of $a+y$.

5. Expand $(a-x)^5$.

Ans. $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.

6. Raise $x+a$ to the seventh power.

Ans. $x^7+7x^6a+21x^5a^2+35x^4a^3+35x^3a^4+21x^2a^5$
$+7xa^6+a^7$.

7. Raise $a+1$ to the third power.

Ans. a^3+3a^2+3a+1.

Note. The powers of 1, being 1, are of course suppressed.

What is Note 1? Explain the operation.

8. Required the sixth power of $1 - x$.

Ans. $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.

9. What is the eighth power of $a + x$?

Ans. $a^8 + 8a^7x + 28a^6x^2 + 56a^5x^3 + 70a^4x^4 + 56a^3x^5 + 28a^2x^6 + 8ax^7 + x^8$.

201. When either or both terms of a binomial have coefficients or exponents, the theorem may still be applied. For, since such terms, on being raised to any power, must have all their factors affected alike, by being enclosed in a parenthesis, they may be treated as a single literal quantity, care being taken after the theorem has been applied to expand the expression obtained.

1. Raise $2a^2 + cd$ to the third power.

OPERATION.

$$\begin{array}{lllll}
1 & +3 & +3 & +1 & (1) \\
(2a^2)^3 & (2a^2)^2 & (2a^2) & & (2) \\
 & (cd) & (cd)^2 & (cd)^3 & (3) \\
\hline
(2a^2)^3 & +3(2a^2)^2(cd) & +3(2a^2)(cd)^2 + & (cd)^3 & (4) \\
8a^6 & + \; 12a^4cd & + \; 6a^2c^2d^2 & + \; c^3d^3 & (5)
\end{array}$$

In (1) we have arranged the coefficients and signs; in (2), $2a^2$ and its exponents; and in (3), cd and its exponents. Combining these, we have (4), which, on being developed, gives (5).

After a little practice, (4) can be written out at once. If any difficulty is experienced in using such terms as $2a^2$ and cd, the formula for the same power may be first written with single letters, such as m and n, and then $(2a^2)$ and (cd) may be substituted in place of those letters.

2. Expand $(3a + 2b)^4$.

Ans. $81a^4 + 216a^3b + 216a^2b^2 + 96ab^3 + 16b^4$.

How may the theorem be applied to binomials, when either or both terms have coefficients or exponents? Explain the operation. What other methods may be used?

3. Required the cube of $2a - 3x$.

Ans. $8a^3 - 36a^2x + 54ax^2 - 27x^3$.

4. Required the fourth power of $1 + 3x$.

Ans. $1 + 12x + 54x^2 + 108x^3 + 81x^4$.

5. Raise $a^2 + b^2$ to the third power.

Ans. $a^6 + 3a^4b^2 + 3a^2b^4 + b^6$.

6. Required the third power of $3x - 5$.

7. Raise $3xy - a$ to the second power.

Ans. $9x^2y^2 - 6axy + a^2$.

8. Required the square of $\frac{1}{2}ab + c$.

Ans. $\frac{1}{4}a^2b^2 + abc + c^2$.

9. Required the second power of $x - \frac{p}{2}$.

Ans. $x^2 - px + \frac{p^2}{4}$.

10. Required the square of $3x + \frac{1}{x}$.

Ans. $9x^2 + 6 + \frac{1}{x^2}$.

11. Required the cube of $a + a^{-1}$.

Ans. $a^3 + 3a + 3a^{-1} + a^{-3}$.

12. Expand $(x^2 + 3y^2)^5$.

Ans. $x^{10} + 15x^8y^2 + 90x^6y^4 + 270x^4y^6 + 405x^2y^8 + 243y^{10}$.

13. Find the third power of $\frac{1}{2}x - \frac{2}{3}y$.

Ans. $\frac{1}{8}x^3 - \frac{1}{2}x^2y + \frac{2}{3}xy^2 - \frac{8}{27}y^3$.

14. Required the sixth power of $x^2 - 2x$.

Ans. $x^{12} - 12x^{11} + 60x^{10} - 160x^9 + 240x^8 - 192x^7 + 64x^6$.

Note. The Binomial Theorem may be applied to the development of the powers of any polynomial whatever. Thus, by changing the form of $a + b + c$ to $(a + b) + c$, or the form of $a + b - c + d$ to $(a + b) - (c - d)$, they may be treated as binomials; and so of any other polynomial.

Repeat the Note.

EVOLUTION.

202. A ROOT of any quantity is a factor taken a certain number of times to form that quantity. Thus,

a is the second or square root of a^2;

a is the third or cube root of a^3.

203. Roots are indicated either by the radical sign, or by a fractional exponent (Art. 22). Thus,

$\sqrt{a}$, or $a^{\frac{1}{2}}$, indicates the second or square root of a;

$\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, indicates the third or cube root of a;

$\sqrt[4]{a}$, or $a^{\frac{1}{4}}$, indicates the fourth root of a;

$\sqrt[n]{a}$, or $a^{\frac{1}{n}}$, indicates the nth root of a;

$\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$, indicates the third root of the second power of a; and so on, the *index* of the radical, or the *denominator* of the fractional exponent, denoting the *degree* of the root.

204. EVOLUTION is the process of extracting any required root of a given quantity. It is the reverse of involution.

205. Any quantity whose root can be extracted is called a *perfect power*, and any quantity whose root cannot be extracted, an *imperfect power*.

A quantity, however, may be a perfect power of one degree, and not of another. Thus,

8 is a perfect cube, but not a perfect square.

Define a Root of any quantity. How are roots indicated? How is the degree of a root denoted? Define Evolution. What is a perfect power? An imperfect power?

206. *The odd roots of a positive quantity are positive.*

For, a positive quantity raised to any power is positive (Art. 187); but a negative quantity raised to any odd power is negative (Art. 188). Thus,

$$\sqrt[3]{a^3} = +a, \text{ or } \sqrt[5]{a^5} = +a.$$

207. *The even roots of a positive quantity are either positive or negative.*

For, either a positive or a negative quantity raised to an even power is positive (Arts. 187, 188). Thus,

$$\sqrt{a^2} = \pm a, \text{ or } \sqrt[4]{a^4} = \pm a.$$

208. *The odd roots of a negative quantity are negative.*

For, a negative quantity raised to an odd power is negative (Art. 188); but a positive quantity raised to any power is positive (Art. 187). Thus,

$$\sqrt[3]{27} = -3, \text{ or } \sqrt[5]{-a^5} = -a.$$

209. *Even roots of a negative quantity are not possible.*

For, no quantity raised to an even power can produce a negative result (Arts. 187, 188). Thus,

$$\sqrt{-4},\ \sqrt[4]{-64}, \text{ and } \sqrt{-a^2},$$

or indicated even roots of negative quantities, are called *impossible*, or *imaginary* quantities.

SQUARE ROOT OF NUMBERS.

210. The SQUARE ROOT, or SECOND ROOT, of a number is a factor which must be taken twice to form that number. Thus,

$$\sqrt{9} = 3, \text{ because } 3 \times 3 = 9.$$

Why are the odd roots of a positive quantity positive? Why are the even roots either positive or negative? Why are the odd roots of a negative quantity negative? Why are the even roots impossible? What are indicated even roots of negative quantities called? Define the Square Root, or second root, of a number.

211. A PERFECT SQUARE is any number or quantity that can be resolved into two equal factors. (Art. 205.)

212. *The square of any integral number consists of twice as many places of figures as the number itself, or of one less than twice as many.*

For, the first ten numbers are

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

and their squares are

1, 4, 9, 16, 25, 36, 49, 64, 81, 100;

also, the square of 99 is 9801, of 100 is 10000, of 999 is 998001, of 1000 is 1000000, and so on. Hence,

213. *If a point be placed over every second figure in any integral number, beginning with the units' place, the number of points will show the number of figures in the square root.*

214. *The square of any number, consisting of more than one place of figures, is equal to the square of the tens, plus twice the product of the tens by the units, plus the square of the units.*

For, if the tens of a number be denoted by a, and the units by b, the number will be denoted by $a + b$, and its square by

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Then, by this formula, if $a = 3$ tens, or 30, and $b = 6$, we have

$$3 \text{ tens} + 6 \text{ units} = 30 + 6 = 36;$$

and $$36^2 = (30 + 6)^2 = 30^2 + 2\,(30 \times 6) + 6^2 = 1296.$$

Again, since every number, consisting of more than one place of figures, may be considered as composed of tens and units, the formula is general, and applies equally whether the root has two places of figures or more than two places. (Nat. Arith., Art. 524.)

Define a Perfect Square. Of how many places of figures does the square of a number consist? How may the number of figures in the square root of a number be shown? To what is the square of any number consisting of more than one place equal?

CASE I.

215. To extract the square root of entire numbers.

1. Let it be required to find the square root of 4356.

OPERATION.

$$\left.\begin{array}{r|l|l} & 4\dot{3}5\dot{6} & 60+6 \\ & 3600 & \\ \hline 120+6 & 756 & \\ & 756 & \\ \hline \end{array}\right\} \text{ or, } \left\{\begin{array}{r|l|l} & 4\dot{3}5\dot{6} & 66 \\ & 36 & \\ \hline 126 & 756 & \\ & 756 & \\ \hline \end{array}\right.$$

By pointing the given number according to Article 213, it appears that the root consists of two places of figures.

Let, now, $a + b$ denote the root, where a is the value of the figure in the tens' place, and b of that in the units' place. Then a must be the greatest multiple of ten whose square is less than 4300; this we find to be 60. Subtracting a^2, that is the square of 60, from the given number, we have the remainder 756, which must contain twice the product of the tens by the units, plus the square of the units, or $2ab + b^2$. Dividing this remainder by $2 \cdot a$, that is by 120, gives 6, which is the value of b. Then $(2a + b)\,b$, that is, 126×6, or 756, is the quantity to be subtracted; and as there is now no remainder, we conclude that $60 + 6$, or 66, is the required square root.

In the work as it stands at the right, the ciphers are omitted.

Had the root consisted of three places of figures, we could have let a represent the hundreds, and b the tens; then, having obtained a and b as before, we might let the hundreds and tens together be considered as a new value of a, and find a new value of b for the units.

RULE.

Separate the given number into periods, by pointing every second figure, beginning with the units' place.

Find the greatest square in the left-hand period, and place its root on the right; subtract the square of this root from the first period, and to the remainder bring down the next period for a dividend.

Explain the operation. Repeat the Rule.

Divide this quantity, omitting the last figure, by double the part of the root already found, and annex the result to the root, and also to the divisor.

Multiply the divisor as it now stands by the part of the root last obtained, and subtract the product from the dividend.

If there are more periods to be brought down, continue the operation in the same manner as before.

NOTE 1. If a root figure is 0, place 0 at the right of the divisor, and bring down the next period to complete the dividend.

NOTE 2. If there be a final remainder, the given number has not an exact root; but we may continue the operation, by annexing an even number of decimal ciphers, and thus obtain a decimal part to be added to the integral part already found.

NOTE 3. In pointing a number having a decimal part, we begin at the units' place, and point both to the right and left of it; and, if the decimal has no exact root, we may continue to form decimal periods to any desirable extent.

NOTE 4. The root of a decimal without an integral part may be found as though the decimal were an entire number, care being taken to make the number of decimal places even, by annexing a cipher, if necessary.

EXAMPLES.

2. Required the square root of 365, to four decimal places.

OPERATION.

```
         365.0000000 0 | 19.1049+
         1
     29 | 265
        | 261
       381 | 400
           | 381
     38204 | 190000
           | 152816
    382089 | 3718400
           | 3438801
               279599
```

What is Note 1? Note 2? Note 3? Note 4? Explain the operation.

It will be observed that four periods of decimal ciphers were annexed to the given number, to correspond with the number of decimal figures required in the root. On obtaining the fourth decimal figure of the root, there is still a remainder, and to show that the root obtained is an approximate one, we annex the sign +.

3. Required the square root of 611524. Ans. 782.

4. Required the square root of 56644. Ans. 238.

5. Required the square root of 6561.

6. Extract the square root of 2116. Ans. 46.

7. Extract the square root of 10246401. Ans. 3201.

8. Extract the square root of 16.2409. Ans. 4.03.

9. Extract the square root of .9409. Ans. .97.

10. What is the square root of .0081? Ans. .09.

11. What is the square root of .006, to four places of decimals? Ans. .0774+.

12. What is the square root of 12, to six places of decimals? Ans. 3.464101+.

13. What is the square root of .0000012321? Ans. .00111.

CASE II.

216. To extract the square root of fractions.

1. Required the square root of $\frac{9}{16}$.

OPERATION.

$$\sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$$

Since to square a fraction we square both its numerator and denominator separately (Art. 191), we find the square root of the given fraction by taking the square root of its terms for the corresponding terms of the root. Hence,

When both terms of a fraction are perfect squares, its square root may be obtained by extracting the square root of both numerator and denominator.

Explain the operation. How is the square root of a fraction obtained when both its terms are perfect squares?

Note. If the fraction has not both terms perfect squares, and cannot be reduced to an equivalent fraction having such terms, its root cannot be exactly found. It may, however, be reduced to a decimal, and the root can then be found as provided in Art. 215, Notes 3 and 4.

2. What is the square root of $\frac{16}{169}$? Ans. $\frac{4}{13}$.
3. What is the square root of $\frac{484}{729}$? Ans. $\frac{22}{27}$.
4. Extract the square root of $2\frac{47}{121}$. Ans. $1\frac{6}{11}$.

Note. Reduce the mixed number to an equivalent common fraction.

5. Required the square root of $\frac{8}{18}$. Ans. $\frac{2}{3}$.

Note. Neither term of the given fraction is a perfect square; but on reducing it to its lowest terms, we obtain $\frac{4}{9}$.

6. Required the square root of $\frac{1899}{10339}$. Ans. $\frac{3}{7}$.
7. Required the square root of $\frac{9}{13}$. Ans. .832 +.
8. Required the square root of $\frac{16}{1031}$.
Ans. .1246, nearly.

CUBE ROOT OF NUMBERS.

217. The Cube Root, or third root, of a number is a factor which must be taken three times to form that number. Thus,

$$\sqrt[3]{27} = 3, \text{ because } 3 \times 3 \times 3 = 27.$$

218. A Perfect Cube is any number or quantity that can be resolved into three equal factors.

219. *The cube of any integral number consists of three times as many places of figures as the number itself, or of one or two less than three times as many.*

How is the square root of a fraction obtained when its terms are not perfect squares? Define Cube Root. A Perfect Cube. Of how many places of figures does the cube of a number consist?

For, the first ten numbers are

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

and their cubes are

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000;

also, the cube of 99 is 970299, of 100 is 1000000, of 999 is 997002999, of 1000 is 1000000000, and so on. Hence,

220. *If a point be placed over every third figure in any integral number, beginning with the units' place, the number of points will show the number of figures in the cube root.*

221. *The cube of any number, consisting of more than one place of figures, is equal to the cube of the tens, plus three times the product of the square of the tens by the units, plus three times the product of the tens by the square of the units, plus the cube of the units.*

For, if the tens of a number be denoted by a, and the units by b, the number will be denoted by $a + b$, and its cube by

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Then, by this formula, if $a = 3$ tens, or 30, and $b = 6$, we have

$$3 \text{ tens} + 6 \text{ units} = 30 + 6 = 36,$$

and

$$36^3 = (30 + 6)^3$$

$$= 30^3 + 3\,(30^2 \times 6) + 3\,(30 \times 6^2) + 6^3 = 46656.$$

Again, since every number, consisting of more than one place of figures, may be considered as composed of tens and units, the formula is general.

CASE I.

222. To extract the cube root of entire numbers.

1. Let it be required to find the cube root of 405224.

How may the number of figures in the cube root of a number be shown? To what is the cube of any number consisting of more than one figure equal?

OPERATION.

$$\begin{array}{r|r|l}
 & 40\dot{5}22\dot{4} & 70+4 \\
 & 343000 & \\ \cline{2-2}
14700 & 62224 & \\
840 & & \\
16 & & \\ \cline{1-1}
15556 \times 4 = & 62224 & \\ \cline{2-2}
\end{array}$$

By pointing the given number according to Article 220, it appears that the root consists of two places of figures.

Let, now, $a + b$ denote the root, where a is the value of the figure in the tens' place, and b of that in the units' place. Then a must be the greatest multiple of ten whose cube is less than 405000; this we find to be 70. Subtracting a^3, that is the cube of 70, or 343000, from the given number, we have the remainder 62224, which must contain three times the product of the square of the tens by the units, plus three times the product of the tens by the square of the units, plus the cube of the units, or $3a^2b + 3ab^2 + b^3$. Dividing this remainder by $3a^2$, that is by three times the square of 70, or 14700, we obtain the value of b, or 4. Then $(3a^2 + 3ab + b^2)b$, that is, $(14700 + 840 + 16)\,4$, or $15556 \times 4 = 62224$, is the quantity to be subtracted; and as there is now no remainder, we conclude that $70 + 4$, or 74, is the required root.

For brevity in the operation, instead of writing 70 in the root, we may simply write 7 in the tens' place, and for the cube of 70 write the cube of 7, or 343, without the ciphers, observing to place the figures under the proper period.

Had the root consisted of three figures, we could have let a represent the hundreds, and b the tens; then, having obtained a and b as before, we might let the hundreds and tens together be considered as a new value of a, and find a new value of b for the units.

From the preceding example we deduce the following

Explain the operation.

RULE.

Separate the given number into periods, by pointing every third figure, beginning with the units' place.

Find the greatest cube in the left-hand period, and place its root on the right; subtract the cube of this root from the first period, and to the remainder bring down the next period for a dividend.

At the left of the dividend write three times the square of the root already found, for a trial divisor; divide the dividend, omitting the last two figures, by it, and write the quotient for the next figure of the root.

Add together the trial divisor, with two ciphers annexed; three times the product of the last root figure by the rest of the root, with one cipher annexed; and the square of the last root figure; and the sum will be the complete divisor.

Multiply the complete divisor by the last root figure, and subtract the product from the dividend.

If there are more periods to bring down, continue the operation in the same manner as before.

NOTE 1. The observations made in Notes 1, 2, 3, and 4, under the rule for the extraction of the square root (Art. 215), are equally applicable to the extraction of the cube root, except that two ciphers must be placed at the right of the trial divisor when it is not contained in its corresponding dividend, and in pointing off decimals each period must contain three places of figures.

NOTE 2. As the trial divisor is necessarily an *incomplete* divisor, it is sometimes found, both in cube and in square root, that after completion it gives a product larger than the dividend. In such a case, the root figure last found is too large, and the one next less must be substituted for it.

EXAMPLES.

2. What is the cube root of 8.144865728?

Repeat the Rule. Repeat Note 1. Repeat Note 2.

OPERATION.

$$\begin{array}{rl|l}
 & & \dot{8}.14\dot{4}86\dot{5}72\dot{8}\ \big|\ 2.012 \\
 & & 8 \\
\hline
120000 & & 144865 \\
600 & & \\
1 & & \\
\overline{120601} & \times 1 = & 120601 \\
\hline
12120300 & & 24264728 \\
12060 & & \\
4 & & \\
\overline{12132364} & \times 2 = & 24264728 \\
\hline
\end{array}$$

It will be observed that, in consequence of the 0 in the root, we annex two additional ciphers to the trial divisor, 1200, and bring down to the corresponding dividend another period.

3. What is the cube root of 941192? Ans. 98.
4. What is the cube root of 389017? Ans. 73.
5. What is the cube root of 37259704? Ans. 334.
6. What is the cube root of 251239591? Ans. 631.
7. What is the cube root of 46268279? Ans. 359.
8. Required the cube root of 1481.544. Ans. 11.4.
9. Required the cube root of .008649. Ans. .2052 +.

CASE II.

223. To extract the cube root of fractions.

1. What is the cube root of $\frac{343}{729}$?

OPERATION.

$$\sqrt[3]{\frac{343}{729}}=\frac{\sqrt[3]{343}}{\sqrt[3]{729}}=\frac{7}{9}$$

Since to cube a fraction we cube both of its terms separately (Art. 191), we find the cube root of the given fraction by taking the cube root of its terms for corresponding terms of the root. Hence,

---

Explain the operation of Example 2. Of Example 1, Case II.

When both terms of a fraction are perfect cubes, its cube root may be obtained by extracting the cube root of both numerator and denominator.

NOTE. If the fraction has not both terms perfect cubes, and cannot be reduced to an equivalent fraction having such terms, it may be reduced to a decimal, and the root found as provided in Art. 222.

2. Find the cube root of $\frac{125}{343}$. Ans. $\frac{5}{7}$.

3. Extract the cube root of $\frac{68921}{59319}$. Ans. $1\frac{2}{39}$.

4. Required the cube root of $\frac{2}{19}$. Ans. .472 +.

ROOTS OF MONOMIALS.

224. The rules for evolution must be deduced from those for involution, for the one is the reverse of the other. (Art. 204.)

1. Let it be required to extract the cube root of $27\,a^6 b^3$.

OPERATION.

$$\sqrt[3]{27\,a^6 b^3} = 27^{\frac{1}{3}} \times a^{\frac{6}{3}} \times b^{\frac{3}{3}} = 3\,a^2 b.$$

Since to cube a monomial we cube the coefficient and multiply the exponent of each of its letters by 3, the exponent of the required power (Art. 190), to find the cube root of the given monomial, we reverse the process, and extract the cube root of its coefficient, 27, and divide the exponents of each of its letters, a and b, by 3. The result, $3\,a^2 b$, being an odd root, is positive (Art. 206).

2. Let it be required to extract the square root of $9\,a^4 b^2$.

OPERATION.

$$\sqrt{9\,a^4 b^2} = 9^{\frac{1}{2}} \times a^{\frac{4}{2}} \times b^{\frac{2}{2}} = \pm\,3\,a^2 b.$$

Since extracting the square root is the reverse of the formation

---

When both terms of a fraction are perfect cubes, how is the cube root obtained? How, when its terms are not perfect cubes? Explain the operation of Example 1. Of Example 2.

of the square, we extract the square root of the coefficient 9, and then divide the exponents of the letters a and b by 2. The result, being an even root of a positive quantity, may be either positive or negative (Art. 207), and therefore is written with the double sign ±.

From these examples is deduced the following

RULE.

Extract the required root of the numerical coefficient, and divide the exponent of each letter by the index of the root.

NOTE 1. Prefix to odd roots of positive quantities +, to odd roots of negative quantities —, and to even roots of positive quantities ±.

NOTE 2. The root of a monomial fraction may be found by extracting the required root of each of its terms separately. Thus,

$$\sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \pm\,\frac{a}{b}, \text{ for } \left(\pm\,\frac{a}{b}\right)^2 = \frac{a^2}{b^2}.$$

EXAMPLES.

3. Find the square root of $16\,x^2$. Ans. $\pm 4\,x$.

4. Find the cube root of $27\,a^3$. Ans. $3\,a$.

5. Find the fourth root of $16\,a^4\,x^8$. Ans. $\pm 2\,a\,x^2$.

NOTE. The fourth root of a quantity is one of its four equal factors, or it is the square root of its square root, since the fourth power of a quantity may be found by squaring its second power.

6. What is the square root of $144\,a^4\,b^2\,c^6$?

7. What is the cube root of $125\,a^6\,x^3$? Ans. $5\,a^2\,x$.

8. What is the fifth root of $-32\,a^{10}\,x^5$? Ans. $-2\,a^2\,x$.

9. What is the square root of $\frac{4\,b^2\,x^4}{9\,a^4\,y^2}$? Ans. $\pm\,\frac{2\,b\,x^2}{3\,a^2\,y}$.

Repeat the Rule. Note 1. How may the root of a monomial fraction be found? What is the fourth root of a quantity?

10. Required the cube root of $\frac{27\,x^3}{729\,y^6}$. Ans. $\frac{x}{3\,y^2}$.

11. Required the square root of $25\,a\,b^3\,c^2$.

Ans. $\pm\,5\,a^{\frac{1}{2}}\,b^{\frac{3}{2}}\,c$.

Note. As we cannot extract the square root of either a or b^3, we *indicate* the division of the exponents by 2. Hence the propriety of indicating roots by fractional exponents (Art. 203).

12. Required the cube root of $-729\,a^{-3}\,b^{-6}$.

Ans. $-9\,a^{-1}\,b^{-2}$.

13. Required the value of $\sqrt[5]{243\,x^5\,y}$. Ans. $3\,x\,y^{\frac{1}{5}}$.

14. Required the value of $(169\,a^n\,b^{-1}\,c^{-2})^{\frac{1}{2}}$.

Ans. $\pm\,13\,a^{\frac{n}{2}}\,b^{-\frac{1}{2}}\,c^{-1}$.

15. Required the seventh root of $\frac{a^7\,b}{128\,x^4\,y^7}$.

Ans. $\frac{a\,b^{\frac{1}{7}}}{2\,x^{\frac{4}{7}}\,y}$.

16. Required the value of $(a^n\,b^m\,c^{2n}\,d^{-3n})^{\frac{1}{n}}$.

Ans. $a\,b^{\frac{m}{n}}\,c^2\,d^{-3}$.

SQUARE ROOT OF POLYNOMIALS.

225. The manner of forming the square of a polynomial must, by reversing the process, lead to the discovery of its root. If we take any binomial, as $a+b$, we have

$$(a+b)^2 = a^2 + 2\,a\,b + b^2;$$

and the last two terms of this expression factored give

$$(2\,a+b)\,b.$$

1. Let us now reverse the involution, and discover

How may we discover the process of finding the square root of a polynomial?

how the root $a+b$ may be derived from the square $a^2+2ab+b^2$.

OPERATION.

$$\begin{array}{r|l|l} & a^2+2ab+b^2 & a+b \\ & a^2 & \\ \cline{2-2} 2a+b & 2ab+b^2 & \\ & 2ab+b^2 & \\ \cline{2-2} \end{array}$$

The square root of the first term, a^2, is a, which is the first term of the required root. Subtracting the square of a from the given polynomial, we have $2ab+b^2$, or $(2a+b)\,b$, for a remainder or dividend. Dividing the first term of the dividend, $2ab$, by $2a$, which is double the first term of the root, we obtain b, the other term of the root, which, connected to $2a$, completes the divisor, $2a+b$. Multiplying this divisor by the last term of the root, b, and subtracting the product, $2ab+b^2$, from the remainder, we have nothing left.

By a like process, a root consisting of more than two terms may be found from its square, since all such roots can be expressed in a binomial form. Thus,

$$a+b+c=(a+b)+c,$$

and its square,

$$a^2+2ab+b^2+2ac+2bc+c^2=(a+b)^2+2\,(a+b)\,c+c^2,$$

which, factored, gives

$$a^2+(2a+b)\,b+(2a+2b+c)\,c.$$

2. Let it next be required to find the square root of $a^2+2ab+b^2+2ac+2bc+c^2$.

OPERATION.

$$\begin{array}{r|l|l} & a^2+2ab+b^2+2ac+2bc+c^2 & a+b+c \\ & a^2 & \\ \cline{2-2} 2a+b & 2ab+b^2 & \\ & 2ab+b^2 & \\ \cline{2-2} 2a+2b+c & 2ac+2bc+c^2 & \\ & 2ac+2bc+c^2 & \\ \cline{2-2} \end{array}$$

Explain the operation of Example 1. How may the process be extended to square roots consisting of more than two terms? Explain the operation of Example 2.

We find $a + b$ of the root as in the preceding example, and have, on subtracting and bringing down the remaining terms, $2ac + 2bc + c^2$, for a remainder or dividend. Dividing the first term of this quantity, $2ac$, by $2a$, which is double the first term of the root, we obtain c, the third term of the root, which, connected to $2a + 2b$, or double the part of the root already found, completes the divisor, $2a + 2b + c$. Multiplying this divisor by the last term of the root, c, and subtracting the product, $2ac + 2bc + c^2$, from the dividend, there is nothing left.

From these examples and illustrations we derive the

RULE.

Arrange the terms according to the powers of some letter.

Find the square root of the first term, write it as the first term of the root, and subtract its square from the given polynomial, by bringing down two or more terms for a dividend.

Divide the first term of the dividend by double the part of the root already found, and annex the result to the root, and also to the divisor.

Multiply the divisor as it now stands by the term of the root last obtained, and subtract the product from the dividend.

If there are other terms remaining, continue the operation in the same manner as before.

Note 1. Since all possible even roots may be either positive or negative (Art. 207), the square root obtained by the rule will remain a root, when all its signs are changed.

Note 2. The fourth root may be obtained by taking the square root of the square root.

Examples.

3. Find the square root of $4x^4 - 12x^3 + 5x^2 + 6x + 1$.

Repeat the Rule. What is Note 1? Note 2?

OPERATION.

$$
\begin{array}{r|l|l}
 & 4x^4 - 12x^3 + 5x^2 + 6x + 1 & 2x^2 - 3x - 1 \\
 & 4x^4 & \\
\hline
4x^2 - 3x & -12x^3 + 5x^2 & \\
 & -12x^3 + 9x^2 & \\
\hline
4x^2 - 6x - 1 & -4x^2 + 6x + 1 & \\
 & -4x^2 + 6x + 1 & \\
\hline
\end{array}
$$

4. Find the square root of $a^4 + 4a^2b + 4b^2$.
Ans. $a^2 + 2b$.

5. What is the square root of $9x^4 - 12x^3 + 16x^2 - 8x + 4$? Ans. $3x^2 - 2x + 2$.

6. What is the square root of $x^2 + 4bx + 4b^2$?

7. What is the square root of $a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + 9b^4$? Ans. $a^2 + 2ab + 3b^2$.

8. Required the square root of $a^4 - 2a^3 + 2a^2 - a + \frac{1}{4}$.
Ans. $a^2 - a + \frac{1}{2}$.

9. What is the square root of $x^4 - 2x^2 + 1$?
Ans. $x^2 - 1$.

10. What is the square root of $a^2 - 2 + a^{-2}$?
Ans. $a - a^{-1}$.

11. What is the square root of $4a^2 - 12ab + 4ax + 9b^2 - 6bx + x^2$? Ans. $2a - 3b + x$.

12. Required the fourth root of $a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4$. Ans. $a + 2b$.

CUBE ROOT OF POLYNOMIALS.

226. An investigation of the formation of a polynomial cube, by reversing the process, must lead to the discovery of its root. If we take any binomial, as $a + b$, we have

---

Explain the operation. In what way may we discover the process of finding the cube root of a polynomial?

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

and the last three terms of this expression factored give

$$(3a^2 + 3ab + b^2)\,b.$$

1. Let us now, by reversing the involution, discover how the root $a+b$ may be derived from the cube $a^3 + 3a^2b + 3ab^2 + b^3$.

OPERATION.

$$\begin{array}{r|l|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 & a+b \\
 & a^3 & \\
\hline
3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 & \\
 & 3a^2b + 3ab^2 + b^3 & \\
\hline
\end{array}$$

The cube root of the first term, a^3, is a, which is the first term of the required root. Subtracting the cube of a from the given polynomial, we have $3a^2b + 3ab^2 + b^3$, or $(3a^2 + 3ab + b^2)\,b$, for a remainder or dividend. Dividing the first term of the dividend by the trial divisor, $3a^2$, which is three times the square of the first term of the root, we obtain b, the other term of the root. Adding, now, to the trial divisor, $3a^2$, three times the product of the first term of the root by the last, and the square of the last term of the root, we have for the complete divisor, $3a^2 + 3ab + b^2$. Multiplying this by b, the last term of the root, and subtracting the product from the dividend, there is no remainder, and the root is obtained.

By a like process, a root of more than two terms may be found from its cube, since all such roots can be expressed in a binomial form. Thus,

$$a + b + c = (a+b) + c,$$

and its cube,

$$a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$$
$$= (a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$$

which, factored, gives

$$a^3 + (3a^2 + 3ab + b^2)\,b + (3a^2 + 6ab + 3b^2 + 3ac + 3bc + c^2)\,c.$$

Explain the operation. How may a cube root consisting of more than two terms be obtained from its power?

2. Let it now be required to derive the cube root $a+b+c$ from its power.

OPERATION.

$$\begin{array}{l|l}
\multicolumn{2}{r}{\underline{|\,a+b+c,\ \text{root.}^*}} \\
\multicolumn{2}{l}{a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3} \\
\multicolumn{2}{l}{a^3} \\
\hline
3a^2+ & 3a^2b+3ab^2+b^3 \\
3ab+b^2 & 3a^2b+3ab^2+b^3 \\
\hline
3a^2+6ab+3b^2+ & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \\
\quad 3ac+3bc+c^2 & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \\
\hline
\end{array}$$

We find $a+b$ of the root as in the preceding example, and have, on subtracting and bringing down the remaining terms, $3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$, for a remainder or dividend. Dividing the first term of the dividend by the first term of the trial divisor, $3a^2$, we obtain c, the third term of the root. Adding together three times the square of the first two terms of the root, which is the trial divisor, three times the product of the first two terms by the third, and the square of the third, we have for the complete divisor, $3a^2+6ab+3b^2+3ac+3bc+c^2$. Multiplying this by c, the last term of the root, and subtracting the product from the dividend, there is no remainder, and the root is obtained.

Hence, we deduce the following

RULE.

Arrange the terms according to the powers of some letter.

Find the cube root of the first term, write it as the first term of the root, and subtract its cube from the given polynomial, by bringing down three or more terms for a dividend.

Take three times the square of the part of the root already

Explain the operation. Repeat the Rule.

* The root is written, in this case, above the power, and the divisors each on two lines, to economize space.

found for the trial divisor, divide the first term of the dividend by it, and write the quotient for the next term of the root.

Add together the trial divisor, three times the product of the first term by the last, and the square of the last, for a complete divisor.

Multiply the complete divisor by the last term of the root, and subtract the product from the dividend.

If there are other terms remaining, form a new dividend, and continue the operation in the same manner as before.

NOTE 1. The terms of each new dividend must be arranged, if necessary, according to the powers of the leading letter of the root.

NOTE 2. If there are three terms in the root, the first two terms must take the place of the first term in obtaining the third. The trial divisor will strictly contain three terms, but only the first need be used, till the divisor is completed.

EXAMPLES.

3. What is the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$?

OPERATION.

$$\begin{array}{r|l|l}
 & x^6 + 6x^5 - 40x^3 + 96x - 64 & x^2 + 2x - 4 \\
 & x^6 & \\ \hline
3x^4 + 6x^3 + 4x^2 & 6x^5 - 40x^3 & \\
 & 6x^5 + 12x^4 + 8x^3 & \\ \hline
3x^4 + 12x^3 - 24x + 16 & -12x^4 - 48x^3 + 96x - 64 & \\
 & -12x^4 - 48x^3 + 96x - 64 & \\ \hline
\end{array}$$

We here bring down only two terms at each time, instead of three, since in the given expression two terms, those containing x^4 and x^2, are wanting. In the last complete divisor, $12x^2$ and $-12x^2$ cancel each other.

4. What is the cube root of $x^3 + 3x^2y + 3xy^2 + y^3$?

Ans. $x + y$.

Repeat Note 1. Note 2.

5. Find the cube root of $y^6 - 3y^5 + 5y^3 - 3y - 1$.
Ans. $y^2 - y - 1$.

6. Find the cube root of $27x^3 + 54x^2y + 36xy^2 + 8y^3$.

7. Find the cube root of $m^6 + 6m^5 - 40m^3 + 96m - 64$.
Ans. $m^2 + 2m - 4$.

8. Required the cube root of $a^3 + 3a + 3a^{-1} + a^{-3}$.
Ans. $a + a^{-1}$.

RADICALS.

227. A Radical is a root of a quantity indicated either by a radical sign or by a fractional exponent; as, $\sqrt{a}$, $a^{\frac{1}{2}}$, and $2\sqrt[3]{1+a}$.

When the root indicated can be exactly obtained, it is called a *rational quantity*, and when it cannot be exactly obtained, it is called an *irrational* or *surd quantity*. Thus, $\sqrt[3]{27a^3}$, which can be expressed by $3a$, is called a rational quantity; and $\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$, is called an irrational or surd quantity.

An even root of a negative quantity cannot be obtained, even approximately, and is therefore called an *imaginary quantity* (Art. 209).

228. The Coefficient of a radical is the quantity or factor prefixed to it. Thus, in $2\sqrt{2bc^3}$, and $a(c+d)^{\frac{1}{3}}$, 2 and a are the coefficients.

229. The Degree of a radical is denoted by the index of the radical sign, or by the denominator of the fractional exponent. Thus,

$\sqrt{a}$, $\sqrt{y}$, $(abc)^{\frac{1}{2}}$, are radicals of the *second* degree;

Define a Radical. When is a quantity called rational? When irrational or surd? When imaginary? Define the Coefficient of a radical. The Degree of a radical.

$\sqrt[3]{x^2}$, $b^{\frac{1}{3}}$, $(2a^2x^3y)^{\frac{1}{3}}$, are radicals of the *third* degree;

$\sqrt[n]{ac}$, $3\sqrt[n]{m}$, $(a+b)^{\frac{1}{n}}$ are radicals of the *n*th degree.

230. SIMILAR RADICALS are those of the same degree, with the same quantity under the radical sign. Thus, $5\sqrt[3]{ax}$, and $7\sqrt[3]{ax}$ are similar radicals; and also $a\,y^{\frac{1}{n}}$ and $c\,y^{\frac{1}{n}}$.

REDUCTION OF RADICALS.

231. REDUCTION OF RADICALS is the process of changing their forms without altering their values.

232. The reduction of radicals depends upon the general principle, that

The root of any quantity is equal to the product of the like roots of its several factors.

For, in obtaining the root of a monomial, we obtain the root of each of its factors, whether numerical or literal (Art. 224).

CASE I.

233. To reduce radicals to their simplest form.

A radical is in its *simplest form*, when it has under the sign no factor which is a perfect power.

1. Reduce $\sqrt[3]{135\,a^6b^4}$ to its simplest form.

OPERATION.

$$\begin{aligned}\sqrt[3]{135\,a^6b^4} &= \sqrt[3]{27\,a^6b^3 \times 5\,b}\\ &= \sqrt[3]{27\,a^6b^3} \times \sqrt[3]{5\,b}\\ &= 3\,a^2b\sqrt[3]{5\,b}\end{aligned}$$

We first resolve the quantity under the radical sign into two factors, one of which, $27\,a^6b^3$, is a perfect cube. Then, since the root of a quantity is equal to the product of the roots of its

---

Define Similar Radicals. Reduction of Radicals. Upon what principle does the reduction of radicals depend? Explain the operation.

several factors (Art. 232), we find the root, $3a^2b$, of the rational part, and multiply it by the indicated root of the surd factor, or, which is the same thing, write it as the coefficient of the surd factor placed under the sign; and thus we obtain $3a^2b\sqrt[3]{5b}$, the simplest form of the radical. Hence the

RULE.

Resolve the quantity under the radical sign into two factors, one of which shall contain all the perfect powers of the same degree as the radical. Extract the required root of this factor, and write it as a coefficient of the other factor, placed under the sign.

EXAMPLES.

Reduce the following radicals to their simplest forms.

2. $\sqrt{9a^4x}$. Ans. $3a^2\sqrt{x}$.

3. $\sqrt{32a^2x}$. Ans. $4a\sqrt{2x}$.

4. $7\sqrt{80x}$. Ans. $28\sqrt{5x}$.

5. $a\sqrt{125b^3}$. Ans. $5ab\sqrt{5b}$.

6. $\sqrt[3]{64a^5b^6}$. Ans. $4ab^2\sqrt[3]{a^2}$.

7. $\sqrt{50ab^2c^2}$.

8. $(ax^3+bx^4)^{\frac{1}{3}}$. Ans. $x(a+bx)^{\frac{1}{3}}$.

9. $2(x^3-a^2x^2)^{\frac{1}{2}}$. Ans. $2x(x-a^2)^{\frac{1}{2}}$.

10. $\sqrt[3]{5(a^3+a^4b)}$. Ans. $a\sqrt[3]{5(1+ab)}$.

11. $6\sqrt{54a^3b^3c}$. Ans. $18ab\sqrt{6abc}$.

12. $3\sqrt[5]{32a^6b^2c^5}$. Ans. $6ac\sqrt[5]{ab^2}$.

13. $(72x+108y)^{\frac{1}{2}}$. Ans. $6(2x+3y)^{\frac{1}{2}}$.

14. $5(a-b)\sqrt{a^2c+2abc+b^2c}$.
Ans. $5(a^2-b^2)\sqrt{c}$.

Repeat the Rule.

234. When the given radical is in a fractional form, it will often be convenient, before applying the rule, *to multiply both terms of the fraction by such a quantity as will make the denominator a perfect power of the degree indicated.* Then the factor under the sign in the simplest form of the radical will be an entire quantity.

1. Reduce $\sqrt[3]{\frac{2}{3}}$ to its simplest form.

OPERATION.

$$\sqrt[3]{\tfrac{2}{3}} = \sqrt[3]{\tfrac{18}{27}} = \sqrt[3]{\tfrac{1}{27} \times 18} = \tfrac{1}{3}\sqrt[3]{18}$$

2. Reduce $3\sqrt{\frac{4a^2}{5}}$ to its simplest form.

OPERATION.

$$3\sqrt{\frac{4a^2}{5}} = 3\sqrt{\frac{20a^2}{25}} = 3\sqrt{\frac{4a^2}{25} \times 5} = 3 \times \frac{2a}{5}\sqrt{5} = \frac{6a}{5}\sqrt{5}$$

Reduce the following radicals to their simplest forms.

3. $2\sqrt{\frac{3}{8}}$. Ans. $\frac{1}{2}\sqrt{6}$.

4. $\frac{3}{4}\sqrt{\frac{2ax^3}{3}}$. Ans. $\frac{x}{4}\sqrt{6ax}$.

5. $\left(\frac{a^4b^3}{4}\right)^{\frac{1}{3}}$. Ans. $\frac{ab}{2}(2a)^{\frac{1}{3}}$.

6. $4\sqrt{\frac{3a^3}{8b}}$. Ans. $\frac{a}{b}\sqrt{6ab}$.

7. $2\left(\frac{5a^5b}{8c^3}\right)^{\frac{1}{4}}$. Ans. $\frac{a}{c}(10abc)^{\frac{1}{4}}$.

CASE II.

235. To reduce a rational quantity to the form of a radical.

1. Reduce $2a^2$ to the form of the cube root.

When the radical is in a fractional form, how may we proceed? Explain the first operation. The second operation.

OPERATION.

$$2a^2 = (2a^2)^{\frac{3}{3}} = (8a^6)^{\frac{1}{3}}$$

Since the radical required is of the third degree, we cube each of the factors of $2a^2$, and obtain $8a^6$, which, written under the sign of the root indicated, gives the required form, or $\sqrt[3]{8a^6}$. The value of this expression is evidently $2a^2$; and in general, since evolution is the reverse of involution, powers and roots of the same degree cancel each other like the terms of fractions. Hence the

RULE.

Raise the quantity to the power indicated by the given root, and write it under the corresponding radical sign.

EXAMPLES.

2. Reduce $3ax$ to the form of the square root.
Ans. $\sqrt{9a^2x^2}$.

3. Reduce $-5a^2b$ to the form of the cube root.
Ans. $\sqrt[3]{-125a^6b^3}$.

4. Reduce $2x - 3$ to a radical of the second degree.
Ans. $(4x^2 - 12x + 9)^{\frac{1}{2}}$.

5. Reduce $2a^2x^3y$ to a radical of the fourth degree.

6. Reduce $\frac{3ax^2}{2by^3}$ to a radical of the fifth degree.
Ans. $\sqrt[5]{\frac{243a^5x^{10}}{32b^5y^{15}}}$.

236. A *coefficient*, or a *factor of a coefficient*, of a radical may be placed under the radical sign, by *raising it to the power indicated by the radical, and multiplying the quantity already under the sign by the result.*

1. Reduce $3a\sqrt{7}$ to a radical without a coefficient.

Explain the first operation under Case II. Repeat the Rule. How may a coefficient, or a factor of a coefficient, be placed under the radical sign?

OPERATION.

$$3a\sqrt{7} = \sqrt{(3a)^2 \times 7} = \sqrt{63a^2}$$

2. Reduce $5\sqrt[3]{xy-1}$ to a radical without a coefficient.
Ans. $\sqrt[3]{125xy-125}$.

3. In the expression $2a^2\sqrt[4]{3ab}$, place the factor 2 under the sign. Ans. $a^2\sqrt[4]{48ab}$.

4. Reduce $(a+b)\sqrt{c}$ to a radical without a coefficient.
Ans. $(a^2c+2abc+b^2c)^{\frac{1}{2}}$.

5. Reduce $\frac{a}{b}\sqrt{\frac{b^2c}{a^2-b^2}}$ to a radical without a coefficient.
Ans. $\sqrt{\frac{a^2b^2c}{a^2b^2-b^4}}$.

CASE III.

237. To reduce radicals of different degrees to equivalent ones having a common index.

1. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$ to a common index.

OPERATION.

$$a^{\frac{1}{2}} = a^{\frac{3}{6}} = (a^3)^{\frac{1}{6}}, \text{ or } \sqrt[6]{a^3}$$
$$b^{\frac{1}{3}} = b^{\frac{2}{6}} = (b^2)^{\frac{1}{6}}, \text{ or } \sqrt[6]{b^2}$$

Since any quantity may be raised to any power indicated by a given root, and written under a corresponding radical sign, without changing the value of the expression (Art. 235), it is evident that $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$ are equal to $a^{\frac{3}{6}}$ and $b^{\frac{2}{6}}$, respectively. That is, we may reduce the given exponents to equivalent ones having a common denominator. Now, $a^{\frac{3}{6}}$ is equal to $(a^3)^{\frac{1}{6}}$, or $\sqrt[6]{a^3}$, and $b^{\frac{2}{6}}$ is equal to $(b^2)^{\frac{1}{6}}$, or $\sqrt[6]{b^2}$. Hence the

RULE.

Reduce the given exponents to a common denominator; raise each quantity to the power denoted by the numerator of the reduced exponent, and indicate the root denoted by the denominator.

Explain the second operation under Case II. Explain the first operation under Case III. Repeat the Rule.

Note. Radicals may be reduced to a common index, without the use of fractional exponents, by multiplying the index and the exponents of each by such a quantity as will make its index the least common multiple of the given indices.

Examples.

2. Reduce $a\sqrt[n]{x}$ and $b\sqrt[m]{y}$ to a common index.

OPERATION.

$$a\sqrt[n]{x} = a x^{\frac{1}{n}} = a x^{\frac{m}{mn}} = a\,(x^m)^{\frac{1}{mn}} = a\sqrt[mn]{x^m}.$$

$$b\sqrt[m]{y} = b\,y^{\frac{1}{m}} = b\,y^{\frac{n}{mn}} = b\,(y^n)^{\frac{1}{mn}} = b\sqrt[mn]{y^n}.$$

3. Reduce $\sqrt[12]{2}$ and $3\sqrt[4]{3}$ to a common index. Ans. $\sqrt[12]{2}$ and $3\sqrt[12]{27}$.

4. Reduce $\sqrt[4]{a}$, $(5\,b)^{\frac{1}{6}}$, and $(a^2+b^2)^{\frac{1}{3}}$ to a common index. Ans. $(a^3)^{\frac{1}{12}}$, $(25\,b^2)^{\frac{1}{12}}$, and $[(a^2+b^2)^4]^{\frac{1}{12}}$.

5. Reduce $\sqrt{a}$, $\sqrt[3]{a-b}$, and $\sqrt[5]{a+b}$ to a common index. Ans. $\sqrt[30]{a^{15}}$, $\sqrt[30]{(a-b)^{10}}$, and $\sqrt[30]{(a+b)^6}$.

ADDITION OF RADICALS.

238. When radicals to be added are similar, the common radical part, with the sum of their coefficients, will constitute the sum of the radicals.

1. Find the sum of $\sqrt{18}$ and $\sqrt{8}$.

OPERATION.

$$\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$$
$$\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$$
$$\text{Sum} = 5\sqrt{2}$$

We reduce the given radicals to their simplest forms, and have those which are similar. Finding, then, $\sqrt{2}$ to be the common radical part, we have 3 times and 2 times $\sqrt{2}$, equal to 5 times $\sqrt{2}$, or $5\sqrt{2}$.

Explain the operation of Example 2. What may constitute the unit of addition when radicals are added? Explain the operation of Example 1.

2. Find the sum of $\sqrt{x^3}$, $\sqrt{4x^3}$, and $\sqrt{a^2x}$.

OPERATION.

$$\begin{aligned} \sqrt{x^3} &= \sqrt{x^2 \times x} = x\sqrt{x} \\ \sqrt{4x^3} &= \sqrt{4x^2 \times x} = 2x\sqrt{x} \\ \sqrt{a^2x} &= \sqrt{a^2 \times x} = a\sqrt{x} \\ \text{Sum} &= (3x + a)\sqrt{x} \end{aligned}$$

Reducing the given radicals to those which are similar, of which $\sqrt{x}$ is the common radical part, we have x times, $2x$ times, and a times $\sqrt{x}$, or $3x + a$ times $\sqrt{x}$, or $(3x + a)\sqrt{x}$.

Hence the following

RULE.

Reduce each radical, if necessary, to its simplest form. If, then, the radicals are similar, add their coefficients, and to the sum annex the common radical; but if they are dissimilar, indicate the addition by the proper sign.

Note. Since dissimilar radicals have no common radical part, it is evident that their addition can *only* be indicated.

Examples.

3. Find the sum of $5\sqrt{98x}$ and $10\sqrt{2x}$.
Ans. $45\sqrt{2x}$.

4. Find the sum of $\sqrt[3]{48a}$ and $\sqrt[3]{162a}$.
Ans. $5\sqrt[3]{6a}$.

5. Find the sum of $\sqrt[4]{32}$ and $5\sqrt[4]{2}$. Ans. $7\sqrt[4]{2}$.

6. Find the sum of $\sqrt{3a^2b}$ and $\sqrt{3x^2b}$.
Ans. $(a + x)\sqrt{3b}$.

7. Find the sum of $5\sqrt{20a^2x}$ and $3\sqrt{45a^2x}$.
Ans. $19a\sqrt{5x}$.

8. What is the sum of $(3a^2b)^{\frac{1}{2}}$ and $(27a^2b)^{\frac{1}{2}}$?
Ans. $4a(3b)^{\frac{1}{2}}$.

9. What is the sum of $(45c^3)^{\frac{1}{2}}$, $(80c^3)^{\frac{1}{2}}$, and $(5a^2c)^{\frac{1}{2}}$?

Explain the operation. Repeat the Rule. Why can the addition of dissimilar radicals be only indicated?

10. What is the sum of $\sqrt[3]{b^4 y}$ and $\sqrt[3]{b y^4}$?
Ans. $(b+y)\sqrt[3]{b y}$.

11. Find the sum of $\sqrt{3}$ and $\sqrt{\frac{1}{3}}$. Ans. $\frac{4}{3}\sqrt{3}$.

Note. $\sqrt{\frac{1}{3}} = \sqrt{\frac{1}{9} \times 3} = \frac{1}{3}\sqrt{3}$; $\sqrt{3} + \frac{1}{3}\sqrt{3} = \frac{4}{3}\sqrt{3}$.

12. Find the sum of $12\sqrt[3]{\frac{1}{4}}$ and $3\sqrt[3]{\frac{1}{32}}$.
Ans. $\frac{27}{4}\sqrt[3]{2}$.

13. Find the sum of $2\sqrt[3]{5}$, $5\sqrt{6a}$, and $\sqrt{4x}$.
Ans. $2\sqrt[3]{5} + 5\sqrt{6a} + 2\sqrt{x}$.

14. Find the sum of $\sqrt{x+1}$ and $\sqrt{4x+4}$.
Ans. $3\sqrt{x+1}$.

15. Find the sum of $(x^3 y)^{\frac{1}{3}}$, $(x y^3)^{\frac{1}{3}}$, and $(8a^4 y)^{\frac{1}{3}}$.
Ans. $x y^{\frac{1}{3}} + x^{\frac{1}{3}} y + 2a(ay)^{\frac{1}{3}}$.

16. Find the sum of $\frac{1}{2}\sqrt{a^2 b c}$ and $\frac{1}{3}\sqrt{4 b c x^4}$.
Ans. $\left(\frac{a}{2} + \frac{2x^2}{3}\right)\sqrt{b c}$.

17. Find the value of $\sqrt[3]{16a^3 b} + \sqrt{4a^2 b} + \sqrt[3]{54a^3 b} + \sqrt{a^2 b}$.
Ans. $5a\sqrt[3]{2b} + 3a\sqrt{b}$.

SUBTRACTION OF RADICALS.

239. When the radicals are similar, the common radical part, with the difference of their coefficients, will constitute the difference of the radicals.

1. Find the difference between $\sqrt{125a}$ and $\sqrt{20a}$.

OPERATION.

$$\begin{aligned} \sqrt{125a} &= \sqrt{25 \times 5a} = 5\sqrt{5a} \\ \sqrt{20a} &= \sqrt{4 \times 5a} = 2\sqrt{5a} \\ \text{Difference} &= 3\sqrt{5a} \end{aligned}$$

Reducing the given radicals to their simplest forms, we have those which are similar. Finding, then, $\sqrt{5a}$ to be the common radical part, we take 2 times $\sqrt{5a}$ from 5 times $\sqrt{5a}$, and have as their difference 3 times $\sqrt{5a}$, or $3\sqrt{5a}$. Hence the

---

When the radicals are similar, what constitutes the unit of subtraction? Explain the operation.

RULE.

Reduce each radical, if necessary, to its simplest form. If then the radicals are similar, subtract the coefficient of the subtrahend from that of the minuend, and to the difference annex the common radical; but if they are dissimilar, indicate the subtraction by the proper sign.

EXAMPLES.

2. From $\sqrt{45\,a}$ take $\sqrt{5\,a}$. Ans. $2\sqrt{5\,a}$.

3. From $\sqrt[3]{192}$ take $\sqrt[3]{24}$. Ans. $2\sqrt[3]{3}$.

4. From $(9\,a^4\,x)^{\frac{1}{2}}$ take $(4\,a^4\,x)^{\frac{1}{2}}$. Ans. $a^2\sqrt{x}$.

5. From $b\sqrt[3]{8\,a^6\,b}$ take $a\sqrt[3]{a^3\,b^4}$. Ans. $a^2\,b\sqrt[3]{b}$.

6. From $4\,(2+y)^{\frac{1}{3}}$ take $3\,(y+2)^{\frac{1}{3}}$. Ans. $(2+y)^{\frac{1}{3}}$.

7. Find the difference between $\sqrt{108\,a\,x^2}$ and $\sqrt{48\,a\,x^2}$.

8. From $\sqrt{\frac{3}{4}}$ take $\sqrt{\frac{1}{3}}$. Ans. $\frac{1}{6}\sqrt{3}$.

9. From $2\sqrt{3\,a^2\,b^2\,c}$ take $\sqrt{5\,a\,b^3}$.
Ans. $2\,a\,b\sqrt{3\,c}-b\sqrt{5\,a\,b}$.

10. From $\sqrt[4]{32\,a}$ take $2\sqrt[3]{40\,a}$.
Ans. $2\,(\sqrt[4]{2\,a}-2\sqrt[3]{5\,a})$.

11. Find the value of $\sqrt[3]{8\,a^3\,b+16\,a^4}-\sqrt[3]{b^4+2\,a\,b^3}$.
Ans. $(2\,a-b)\sqrt[3]{2\,a+b}$.

MULTIPLICATION OF RADICALS.

240. The multiplication of radicals depends upon the principle, that

The product of like roots of two quantities is equal to the same root of their product.

Repeat the Rule. Upon what principle does the multiplication of radicals depend?

To prove the principle, let a and b be any two quantities.

Then, by Art. 232, $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$,

Whence, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

1. Multiply $4a\sqrt{2by}$ by $3b\sqrt{2bx}$.

OPERATION.

$$\begin{aligned} 4a\sqrt{2by} \times 3b\sqrt{2bx} &= 4a \times 3b\sqrt{2by \times 2bx} \\ &= 12ab\sqrt{4b^2xy} \\ &= 12ab \times 2b\sqrt{xy} \\ &= 24ab^2\sqrt{xy} \end{aligned}$$

Since it is immaterial in what order the factors are taken (Art. 58), we multiply together the coefficients $4a$ and $3b$, and obtain $12ab$; and then the radical parts $\sqrt{2by}$ and $\sqrt{2bx}$, and, in accordance with the above principle, obtain $\sqrt{4b^2xy}$; or, for the whole product, $12ab\sqrt{4b^2xy}$, which, reduced to its simplest form (Art. 233), is $24ab^2\sqrt{xy}$.

2. Multiply $3\sqrt{2a}$ by $2\sqrt[3]{3a}$.

OPERATION.

$$\begin{aligned} 3\sqrt{2a} \times 2\sqrt[3]{3a} = 3\sqrt[6]{2^3a^3} \times 2\sqrt[6]{3^2a^2} &= 3 \times 2\sqrt[6]{2^3a^3 \times 3^2a^2} \\ &= 6\sqrt[6]{72a^5} \end{aligned}$$

We reduce the given radicals to equivalent ones having a common index (Art. 237), and then multiplying, in the same manner as in the last example, obtain $3 \times 2\sqrt[6]{2^3a^3 \times 3^2a^2}$, which, reduced to its simplest form, is $6\sqrt[6]{72a^5}$.

Hence we deduce the following

RULE.

Reduce the radical parts, if necessary, to a common index; then multiply the coefficients together for the coefficient of the product, and the parts under the radicals for the radical part.

Explain the first operation. The second operation. Repeat the Rule.

EXAMPLES.

3. Multiply $3\sqrt{b}$ by $5\sqrt{x}$. Ans. $15\sqrt{bx}$.

4. Multiply $6\sqrt{54}$ by $3\sqrt{2}$.

5. Multiply $7\sqrt{axy}$ by $3\sqrt{2ax}$.
Ans. $21ax\sqrt{2y}$.

6. Multiply $a\sqrt[m]{x}$ by $b\sqrt[n]{y}$. Ans. $ab\sqrt[mn]{x^n y^m}$.

7. Multiply $4\sqrt[3]{ax}$ by $3\sqrt{xy}$. Ans. $12\sqrt[6]{a^2x^5y^3}$.

8. Multiply $\frac{1}{4}\sqrt{6}$ by $\frac{2}{15}\sqrt{9}$. Ans. $\frac{1}{10}\sqrt{6}$.

9. Multiply $2\sqrt[3]{\frac{2}{3}}$ by $3\sqrt[3]{\frac{5}{6}}$. Ans. $2\sqrt[3]{15}$.

10. Multiply $3b^{\frac{1}{3}}$ by $4a^{\frac{1}{4}}$. Ans. $12\sqrt[12]{a^3b^4}$.

11. Multiply $3a\sqrt[4]{8a^2}$ by $2b\sqrt[4]{4a^2c}$.
Ans. $12a^2b\sqrt[4]{2c}$.

12. Multiply $(a+b)^{\frac{1}{2}}$ by $(a+b)^{\frac{2}{3}}$. Ans. $(a+b)^{\frac{7}{6}}$.

13. Required the product of $\sqrt{\frac{2ax}{3c}}$ by $\sqrt{\frac{9ab}{2x}}$.
Ans. $\sqrt{\frac{3a^2b}{c}}$.

14. Required the product of $\sqrt[3]{6a^{\frac{1}{2}}bc^{-1}}$ by $\sqrt[3]{3^{-1}a^{-\frac{1}{2}}bc^2}$.
Ans. $\sqrt[3]{2b^2c}$.

241. When either or both of the radicals are connected with other quantities by the sign + or —, *each term of the multiplicand must be multiplied by each term of the multiplier.* (Art. 64.)

1. Multiply $a+2\sqrt{b}$ by $a-\sqrt{b}$.

When the radicals are connected with other quantities by + or —, how do we multiply?

FIRST OPERATION.	SECOND OPERATION.
$a + 2\sqrt{b}$	$a + 2b^{\frac{1}{2}}$
$\underline{a - \sqrt{b}}$	$\underline{a - b^{\frac{1}{2}}}$
$a^2 + 2a\sqrt{b}$	$a^2 + 2ab^{\frac{1}{2}}$
$\underline{\quad - a\sqrt{b} - 2\sqrt{b^2}}$	$\underline{\quad - ab^{\frac{1}{2}} - 2b^{\frac{2}{2}}}$
$a^2 + a\sqrt{b} - 2b$	$a^2 + ab^{\frac{1}{2}} - 2b$

It will be seen that both operations give precisely the same result.

From this and previous examples, it may be inferred that, *in multiplication of radicals expressed by fractional exponents, the same rules apply as when the exponents are integral.* If fractional exponents having different denominators are to be added, they must of course be reduced to a common denominator, and this is precisely the same process as reducing radicals to a common index.

When fractional exponents are used, it is often most convenient to allow each factor to take a separate one; but when the radical sign is used, it is most convenient to employ a common one for all the irrational factors of any given term.

2. Multiply $4 + 2\sqrt{2}$ by $2 - \sqrt{2}$. Ans. 4.

3. Multiply $(a + x)^{\frac{2}{3}}$ by $(a - x)^{\frac{2}{3}}$. Ans. $(a^2 - x^2)^{\frac{2}{3}}$.

4. Multiply $\sqrt{a + b}$ by $\sqrt{a + b}$. Ans. $a + b$.

5. Required the product of $\frac{1}{2} + \frac{1}{2}\sqrt{5}$ by $\frac{3}{4} + \frac{1}{4}\sqrt{5}$.
Ans. $1 + \frac{1}{2}\sqrt{5}$.

6. Required the product of $\sqrt{x} + \sqrt{a + x}$ by $\sqrt{a + x}$.
Ans. $\sqrt{ax + x^2} + a + x$.

7. Required the product of $a^{\frac{3}{2}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{2}} - 2b^{\frac{3}{2}}$.
Ans. $a^2 + a^{\frac{1}{2}}b^{\frac{2}{3}} - 2a^{\frac{3}{2}}b^{\frac{3}{2}} - 2b^{\frac{13}{6}}$.

What is said of multiplication by fractional exponents? e/

DIVISION OF RADICALS.

242. Division of radicals depends upon the principle that

The quotient of like roots of two quantities is equal to the same root of their quotient.

To prove this principle, let a and b represent two quantities.

Then, by Art. 240, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$,

Whence, $\sqrt{ab} \div \sqrt{a} = \sqrt{\frac{ab}{a}} = \sqrt{b}$.

1. Divide $6\sqrt{54a}$ by $3\sqrt{2a}$.

OPERATION.

$$\frac{6\sqrt{54a}}{3\sqrt{2a}} = \frac{6}{3}\sqrt{\frac{54a}{2a}}$$
$$= 2\sqrt{27}$$
$$= 6\sqrt{3}$$

We divide the coefficient, 6, of the dividend by the coefficient, 3, of the divisor, and obtain 2, and the radical part of the dividend by that of the divisor, and, in accordance with the above principle, obtain $\sqrt{27}$; or, for the whole quotient, $2\sqrt{27}$, which, reduced to its simplest form, is $6\sqrt{3}$.

2. Divide $16\sqrt[3]{a^2}$ by $8\sqrt{a}$.

OPERATION.

$$\frac{16\sqrt[3]{a^2}}{8\sqrt{a}} = \frac{16\sqrt[6]{a^4}}{8\sqrt[6]{a^3}} = 2\sqrt[6]{a}$$

We reduce the given radicals to equivalent ones having a common index (Art. 237), and, dividing in the same manner as in the preceding example, obtain $2\sqrt[6]{a}$.

Hence the following

RULE.

Reduce the radical parts, if necessary, to a common index; then divide the coefficient of the dividend by the coefficient of the divisor, and the radical part of the dividend by that of the

---

Upon what principle does division of radicals depend? Explain the first operation. The second. Repeat the Rule.

divisor, and prefix the first quotient to the last written under the common index.

Examples.

3. Divide $\sqrt{40}$ by $\sqrt{2}$. Ans. $2\sqrt{5}$.

4. Divide $a^{\frac{1}{n}}$ by $b^{\frac{1}{n}}$. Ans. $\sqrt[n]{\frac{a}{b}}$.

5. Divide $\sqrt[3]{135}$ by $\sqrt[3]{5}$.

6. Divide $4\sqrt{a^3}$ by $2\sqrt[4]{a}$. Ans. $2a\sqrt[4]{a}$.

7. Divide $4\sqrt[3]{ax}$ by $3\sqrt{xy}$. Ans. $\frac{4}{3}\sqrt[6]{\frac{a^2}{xy^3}}$.

8. Divide $bc\sqrt[n]{ab}$ by $b\sqrt[n]{a}$. Ans. $c\sqrt[n]{b}$.

9. Required the quotient of $(1-x^2)^{\frac{1}{2}}$ divided by $(1+x)^{\frac{1}{2}}$. Ans. $(1-x)^{\frac{1}{2}}$.

10. Required the quotient of $\sqrt{\frac{a}{b}}$ divided by $\sqrt{\frac{c}{d}}$.

Ans. $\sqrt{\frac{ad}{bc}}$.

11. Required the quotient of $\frac{1}{2}\sqrt[3]{\frac{1}{2}}$ divided by $\frac{1}{3}\sqrt[3]{\frac{1}{3}}$.

Ans. $\frac{3}{4}\sqrt[3]{12}$.

12. Required the value of $(\sqrt{72}+\sqrt{32-4}) \div \sqrt{8}$.

Ans. $3+\frac{1}{2}\sqrt{14}$.

13. Required the quotient of $m\sqrt{\frac{a-b}{a+b}}$ divided by $n\sqrt{\frac{a-b}{a+b}}$. Ans. $\frac{m}{n}$.

14. Required the quotient of $\sqrt{a^2-b^2}$ divided by $a-b$.

Ans. $\sqrt{\frac{a+b}{a-b}}$.

The remarks already made (Art. 241) respecting fractional exponents will apply also to the division of radicals.

15. Divide $a^2+ab^{\frac{1}{2}}-6b$ by $a-2b^{\frac{1}{2}}$.

Ans. $a+3b^{\frac{1}{2}}$.

16. Divide $a^2 + 2a^{\frac{1}{2}}b^{\frac{2}{3}} - 4a^{\frac{3}{2}}b^{\frac{1}{2}} - 8b^{\frac{7}{6}}$ by $a^{\frac{1}{2}} - 4b^{\frac{1}{2}}$.
Ans. $a^{\frac{3}{2}} + 2b^{\frac{2}{3}}$.

INVOLUTION OF RADICALS.

243. Involution of radicals depends upon the same general principles as involution of rational quantities.

1. Raise $2\sqrt{a}$ to the third power.

OPERATION.

$$2\sqrt{a} \times 2\sqrt{a} \times 2\sqrt{a} = 8\sqrt{a^3} = 8a\sqrt{a}.$$

By the definition of involution (Art. 186) we take the given quantity, $2\sqrt{a}$, three times as a factor, and performing the multiplication (Art. 240), we obtain $8\sqrt{a^3}$. This, reduced to its simplest form (Art. 233), gives $8a\sqrt{a}$.

2. Raise $3a + \sqrt{y}$ to the second power.

OPERATION.

$$\begin{array}{lll} 3a & +\sqrt{y} & \\ 3a & +\sqrt{y} & \\ \hline 9a^2 & +3a\sqrt{y} & \\ & 3a\sqrt{y} & +y \\ \hline 9a^2 & +6a\sqrt{y} & +y \end{array}$$

Since the second power is required, we take the given quantity twice as a factor, and, it being a polynomial, we perform the multiplication as in Art. 241.

Hence the following

RULE.

Raise the rational part of a monomial to the required power, and annex the required power of the radical part, written under the given sign.

If the radical part is connected with other quantities by + or —, perform the involution by multiplication of the several terms, as in the multiplication of polynomials.

Explain the first operation. The second. Repeat the Rule.

Note 1. When a quantity is affected by a fractional exponent, its involution may be performed by multiplying that exponent by the exponent of the required power (Art. 190).

Note 2. The result should be reduced to its simplest form. Any factor common to the index of the given radical and the exponent of the required power should be canceled (Art. 235). Hence, when the given radical and the required power are of the same degree, the involution may be performed by removing the radical sign.

EXAMPLES.

3. Required the square of $5\,a^{\frac{1}{3}}$. Ans. $25\,a^{\frac{2}{3}}$.

4. Required the cube of $5\,a\sqrt[3]{y}$. Ans. $125\,a^3\,y$.

5. Raise $x^2\sqrt[3]{6}$ to the second power. Ans. $x^4\sqrt[3]{36}$.

6. Raise $4\,a^2\sqrt{3\,c}$ to the fourth power.

7. Raise $3\sqrt[4]{25\,a\,x}$ to the second power. Ans. $45\sqrt{a\,x}$.

8. Required the fourth power of $\sqrt{3}-\sqrt{2}$. Ans. $49-20\sqrt{6}$.

9. Raise $\sqrt[m]{2\,a}$ to the nth power. Ans. $\sqrt[m]{2^n\,a^n}$.

10. Required the square of $\sqrt{3}+x\sqrt{3}$. Ans. $3+6\,x+3\,x^2$.

11. Required the square of $x^{\frac{1}{2}}-y^{-\frac{2}{3}}$. Ans. $x-2\,x^{\frac{1}{2}}\,y^{-\frac{2}{3}}+y^{-\frac{4}{3}}$.

EVOLUTION OF RADICALS.

244. Evolution of radicals depends upon the same general principles as evolution of rational quantities.

1. Find the cube root of $8\sqrt{a^3}$.

Repeat Note 1. Note 2.

OPERATION.

$\sqrt[3]{8\sqrt{a^3}} = \sqrt[3]{8} \times \sqrt[3]{\sqrt{a^3}} = 2\sqrt{a}$

Since the coefficient 8 is a perfect cube, we have for its cube root 2, and the quantity under the sign, a^3, being also a perfect cube, we have for its cube root a; hence the entire root is $2\sqrt{a}$. (Art. 232.)

2. Find the square root of $20\sqrt{5a}$.

OPERATION.

$$\sqrt{20\sqrt{5a}} = \sqrt{4 \times 5\sqrt{5a}} = \sqrt{4} \times \sqrt{5\sqrt{5a}} = 2\sqrt[4]{125a}.$$

The coefficient 20 is not a perfect square, but is composed of two factors, 4 and 5, of which 4 is a perfect square. The square root of 4 is 2, which is the coefficient of the required root. As we cannot take the square root of 5, we square it and introduce it as a factor under the sign. As the quantity under the radical sign is not a perfect square, we denote its root by multiplying the index of the sign by the index of the required root, and we then have as the entire result, $2\sqrt[4]{125a}$.

Hence the following

RULE.

Extract the required root of the rational part of a monomial, if it is a complete power of the required degree, otherwise introduce it under the radical sign.

Extract the required root of the quantity under the radical sign, if it is a complete power of the required degree, otherwise multiply the index of the radical by the index of the required root.

NOTE. When a quantity is affected by a fractional exponent, its evolution may be performed by dividing this exponent by the index of the required root (Art. 224).

Explain the first operation. The second operation. Repeat the Rule. The Note.

EXAMPLES.

3. Find the cube root of $\sqrt{a b^3}$. Ans. $\sqrt[6]{a b^3}$.

4. Find the square root of $4 \sqrt[3]{5 c}$. Ans. $2 \sqrt[6]{5 c}$.

5. Find the cube root of $a^{-1} (x y)^{\frac{1}{2}}$. Ans. $a^{-\frac{1}{3}} x^{\frac{1}{6}} y^{\frac{1}{6}}$.

6. Find the square root of $25 \sqrt[3]{4 a^2 b}$.

7. Find the sixth root of $a \sqrt{a}$. Ans. $\sqrt[4]{a}$.

8. Required the cube root of $\frac{a}{3} \sqrt{\frac{a}{3}}$. Ans. $\frac{1}{3} \sqrt{3 a}$.

9. Required the cube root of $125 x^{\frac{1}{2}}$. Ans. $5 x^{\frac{1}{6}}$.

10. Extract the fourth root of $64 a^8 b^4 \sqrt{2 c d}$.
Ans. $2 a^2 b \sqrt[8]{32 c d}$.

11. Extract the square root of $x^{\frac{4}{3}} + 6 x^{\frac{2}{3}} y^{\frac{1}{2}} + 9 y$, by means of the rule found in Art. 225. Ans. $x^{\frac{2}{3}} + 3 y^{\frac{1}{2}}$.

RATIONALIZATION.

245. RATIONALIZATION is the process of removing the radical sign from a quantity by multiplication.

It is often necessary to transform a fraction having an irrational denominator, into one whose denominator is *rational*.

CASE I.

246. To rationalize any monomial surd.

1. Rationalize $\sqrt{a}$.

Define Rationalization.

OPERATION.

$$\sqrt{a} \times \sqrt{a} = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a$$

We multiply the given radical, $\sqrt{a}$, by the same quantity, with such an index as will make the sum of the corresponding fractional exponents of the two quantities equal to unity.

2. Rationalize $a^{\frac{2}{5}}$.

OPERATION.

$$a^{\frac{2}{5}} \times a^{\frac{3}{5}} = a$$

We multiply the given radical, $a^{\frac{2}{5}}$, by the same quantity with such a fractional exponent as will make the sum of the fractional exponents of the two quantities equal to unity. Hence the

RULE.

Multiply the given surd by the same quantity with such a fractional exponent as, when added to the fractional exponent of the given quantity, shall be equal to unity.

EXAMPLES.

3. What factor will rationalize $x^{\frac{3}{4}}$? Ans. $x^{\frac{1}{4}}$.

4. What factor will rationalize $4\sqrt[3]{a b^2}$? Ans. $\sqrt[3]{a^2 b}$.

5. What factor will rationalize $a^{-\frac{1}{2}}$, and at the same time make its exponent positive? Ans. $a^{\frac{3}{2}}$.

CASE II.

247. To rationalize a binomial surd containing only the square root.

1. Rationalize $\sqrt{a} + \sqrt{b}$.

Explain the first operation. The second. Repeat the Rule.

OPERATION.

$$\begin{array}{rr}
\sqrt{a}+\sqrt{b} & \\
\sqrt{a}-\sqrt{b} & \\
\hline
a+\sqrt{ab} & \\
-\sqrt{ab} & -b \\
\hline
a & -b
\end{array}$$

Since the product of the sum and difference of two quantities is equal to the difference of their squares (Theo. III. Art. 78), we multiply the given binomial by the same terms, with one of the signs changed, and obtain $a-b$, a rational quantity. Hence the

RULE.

Multiply the given binomial by the same terms, with one of the signs changed.

EXAMPLES.

2. Rationalize $a+\sqrt{b}$. Ans. a^2-b.

3. Rationalize $\sqrt{5}-\sqrt{1}$. Ans. $5-1$, or 4.

248. A *trinomial* surd may be reduced to a binomial surd by multiplying it by the same terms, with the sign of one of them changed, and then the binomial may be rationalized.

Thus, to rationalize $\sqrt{7}+\sqrt{3}-\sqrt{2}$, we have

$$(\sqrt{7}+\sqrt{3}-\sqrt{2})\,(\sqrt{7}+\sqrt{3}+\sqrt{2})=8+2\sqrt{21},$$

and then

$$(8+2\sqrt{21})\,(-8+2\sqrt{21})=84-64=20.$$

CASE III.

249. To rationalize either of the terms of a fractional surd.

1. Reduce $\frac{a}{\sqrt{b}}$ to a fraction whose denominator shall be rational.

Explain the operation. Repeat the Rule. How may a trinomial surd be rationalized?

OPERATION.

$$\frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

We multiply both terms of the fraction by $\sqrt{b}$, and it becomes $\frac{a\sqrt{b}}{b}$, in which the denominator is rational, and the value of the fraction is not changed. Hence the

RULE.

Multiply both numerator and denominator by a factor that will render either of them rational, as may be required.

EXAMPLES.

2. Reduce $\frac{\sqrt{a}}{\sqrt{b}}$ to a fraction having a rational numerator.

Ans. $\frac{a}{\sqrt{ab}}$.

3. Reduce $\frac{1}{\sqrt{3}+1}$ to a fraction having a rational denominator.

Ans. $\frac{\sqrt{3}-1}{2}$.

4. Reduce $\frac{2}{\sqrt{3}}$ to a fraction having a rational denominator.

5. Reduce $\frac{a}{\sqrt{b}+\sqrt{c}}$ to a fraction having a rational denominator.

Ans. $\frac{a(\sqrt{b}-\sqrt{c})}{b-c}$.

6. Reduce $\frac{\sqrt{2}}{3-\sqrt{2}}$ to a fraction having a rational denominator.

Ans. $\frac{3\sqrt{2}+2}{7}$.

7. Reduce $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$ to a fraction having a rational denominator.

Ans. $\frac{(\sqrt{x}+\sqrt{y})^2}{x-y}$.

Explain the operation. Repeat the Rule.

IMAGINARY QUANTITIES.

250. An IMAGINARY QUANTITY is an indicated even root of a negative quantity. Other quantities, whether rational or irrational, are called *real.*

Although this root of a negative quantity is a symbol of an impossible operation (Art. 209), yet it is not without its use in mathematical analysis.

251. *Every imaginary quantity can be resolved into two factors, one of which is real, and the other the imaginary expression,* $\sqrt{-1}$, or $\sqrt[2n]{-1}$.

For, let a denote any real quantity, and $\sqrt{-a}$ any imaginary quantity of the second degree.

Then, $\sqrt{-a} = \sqrt{a\ (-1)} = \pm \sqrt{a} \times \sqrt{-1}$;

also, $\sqrt{-a^2} = \sqrt{a^2\,(-1)} = \pm \sqrt{a^2} \times \sqrt{-1} = \pm a \sqrt{-1}$,

and so on.

Hence, $\sqrt{-1}$, or $\sqrt[2n]{-1}$, may be regarded as a universal factor of every imaginary quantity, and, consequently, may be used as the only symbol of such a quantity.

252. In the addition and subtraction of imaginary quantities, the operations are the same as for other radicals; but with regard to their multiplication and division, the rules for common radicals require some modification.

253. *The product of two imaginary terms is real, with the sign before the radical as by the common rule reversed.*

For, if we take the product of two imaginary quantities in which the imaginary parts are *equal*, it is evident that the sign of the product is changed by removing the radical. Thus,

$$b \sqrt{-a} \times c \sqrt{-a} = b c\ (-a) = -a b c.$$

Define an Imaginary Quantity. Into what two factors may an imaginary quantity be resolved? Give the illustration and inference. What is said of the addition and subtraction of imaginary quantities? What is said of the product of two imaginary terms?

But, if we take two unequal imaginary quantities, $\sqrt{-a}$ and $\sqrt{-b}$, by the common rule (Art. 240), we have

$$(\sqrt{-a})\ (\sqrt{-b}) = \sqrt{ab}.$$

Now, since the quantity whose root is to be extracted was not produced by that root, but from two *unequal* factors, it does not immediately appear whether the result obtained is to be taken *positively* or *negatively*. We may, however, resolve the imaginary quantity into two factors, of which one is $\sqrt{-1}$ (Art. 251). Then we have

$$\begin{aligned}(+\sqrt{-a})\ (+\sqrt{-b}) &= (+\sqrt{a} \times \sqrt{-1})\ (+\sqrt{b} \times \sqrt{-1})\\ &= +\sqrt{ab} \times (\sqrt{-1})^2\\ &= +\sqrt{ab} \times (-1)\\ &= -\sqrt{ab}.\end{aligned}$$

Hence it appears that the result is properly negative.

In like manner it may be shown, that

$$(-\sqrt{-a})\ (-\sqrt{-b}) = -\sqrt{ab},$$

and $$(+\sqrt{-a})\ (-\sqrt{-b}) = +\sqrt{ab}.$$

Whence, also, it appears that

Like signs produce —, and unlike signs +.

254. *The quotient of one imaginary quantity divided by another is real, with the sign before the radical as by the common rule.*

For, $$\frac{+\sqrt{-ab}}{+\sqrt{-a}} = \frac{+\sqrt{ab} \times \sqrt{-1}}{+\sqrt{a} \times \sqrt{-1}} = +\sqrt{b},$$

and $$\frac{-\sqrt{-ab}}{+\sqrt{-a}} = \frac{-\sqrt{ab} \times \sqrt{-1}}{+\sqrt{a} \times \sqrt{-1}} = -\sqrt{b}.$$

Whence it appears that

Like signs produce +, and unlike signs —.

255. To show the application of the foregoing principles, there are introduced the following

---

What is said of the quotient of one imaginary quantity divided by another?

EXAMPLES.

1. Multiply $\sqrt{-9}$ by $\sqrt{-4}$. Ans. $-\sqrt{36} = -6$.
2. Multiply $2\sqrt{-3}$ by $3\sqrt{-2}$. Ans. $-6\sqrt{6}$.
3. Multiply $1+\sqrt{-1}$ by $1-\sqrt{-1}$. Ans. 2.
4. Divide $a\sqrt{-1}$ by $b\sqrt{-1}$. Ans. $\frac{a}{b}$.
5. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$. Ans. $\frac{3}{2}\sqrt{3}$.
6. Divide $2\sqrt{-10}$ by $-5\sqrt{-2}$. Ans. $-\frac{2}{5}\sqrt{5}$.

RADICAL EQUATIONS

LEADING TO SIMPLE EQUATIONS.

256. RADICAL EQUATIONS are those containing radical quantities; as,

$$\sqrt{3x}+4=5, \text{ or } (4+x)^{\frac{1}{3}}=5.$$

257. The *solution* of a radical equation consists in rationalizing the terms containing the unknown quantity, and in determining its value. Only those which reduce to *simple equations* can be considered here.

More will depend upon the ingenuity of the learner than upon any rules that can be given.

1. Given $\sqrt{x}-2=3$, to find the value of x.

OPERATION.

$$\sqrt{x}-2=3$$

Transposing and uniting, $\sqrt{x} = 5$

Squaring (Art. 151), $x = 25$.

2. Given $\sqrt[4]{11+\sqrt{5x}}=3$, to find the value of x.

Define Radical Equations. In what does the solution of a radical equation consist? Explain the first operation.

OPERATION.

$$\sqrt[4]{11+\sqrt{5x}}=3$$

Involving to the fourth power, $11+\sqrt{5x}=81$

Transposing and uniting, $\sqrt{5x}=70$

Squaring, $5x=4900$

Whence, $x=980.$

3. Given $\sqrt{ax}=\sqrt{a}+\sqrt{x}$, to find the value of x.

OPERATION.

$$\sqrt{ax}=\sqrt{a}+\sqrt{x}$$

Transposing, $\sqrt{ax}-\sqrt{x}=\sqrt{a}$

Factoring, $\sqrt{x}(\sqrt{a}-1)=\sqrt{a}$

Squaring, $x(\sqrt{a}-1)^2=a$

Whence, $x=\dfrac{a}{(\sqrt{a}-1)^2}.$

From the foregoing illustrations are deduced the following general directions: —

1. *Transpose all the terms so that a radical expression may stand on one side of the equation, and the rest of the terms on the other side; then involve each side to a power of the same degree as the radical.*

2. *If there is still a radical expression remaining, the process of involution must be repeated.*

3. *Simplify the equation as much as possible before performing the involution.*

NOTE. Radicals may sometimes be removed by multiplying or dividing both members of an equation by a radical expression; hence they sometimes disappear on clearing of fractions. It is also occasionally convenient to rationalize the denominator of a fraction before removing denominators or involving.

Explain the second operation. The third. Repeat the general directions. The Note.

EXAMPLES.

4. Given $\sqrt{x+1}-2=3$, to find the value of x.
Ans. $x=24$.

5. Given $\sqrt{x+7}=\sqrt{x}+1$, to find the value of x.
Ans. $x=9$.

6. Given $2+\sqrt{3x-30}=5$, to find the value of x.
Ans. $x=13$.

7. Given $\sqrt{3-x}+6=8-1$, to find the value of x.

8. Given $x^{\frac{1}{2}}-7=-3$, to find the value of x.
Ans. $x=16$.

9. Given $\sqrt{x+6}=3$, to find the value of x.
Ans. $x=3$.

10. Given $\sqrt{4+x}=4-\sqrt{x}$, to find the value of x.
Ans. $x=2\frac{1}{4}$.

11. Given $y^{\frac{1}{3}}+5=11$, to find the value of y.
Ans. $y=216$.

12. Given $\sqrt[4]{20-\sqrt{2x}}-2=0$, to find the value of x.
Ans. $x=8$.

13. Given $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$, to find the value of x.
Ans. $x=\frac{1}{1-a}$.

14. Given $x\,(x^{\frac{1}{2}}+x^{-\frac{1}{2}})=a\,x^{\frac{1}{2}}$, to find the value of x.
Ans. $x=a-1$.

15. Given $\frac{\sqrt{x}+3}{3}+3=\frac{3\sqrt{x}}{\sqrt{x}-3}$, to find the value of x.
Ans. $x=36$.

16. Given $\frac{ax-1}{\sqrt{ax}+1}=4+\frac{\sqrt{ax}-1}{2}$, to find the value of x.
Ans. $x=\frac{81}{a}$.

Note. Perform the division indicated in the first member (Art. 78).

QUADRATIC EQUATIONS.

258. A Quadratic Equation is an equation of the *second degree* (Art. 145), or one in which the *square* is the highest power of the unknown quantity; as,

$$ax^2 = b,\ \ x^2 + 8x = 20,\ \ \text{or } a^3x^2 - b^4x = c^5.$$

259. A Cubic Equation is an equation of the *third degree;* as,

$$ax^3 = b,\ \ x^3 + x^2 = 12,\ \ \text{or } ax^3 - bx^2 + cx = d.$$

260. A Biquadratic Equation is an equation of the *fourth degree;* as,

$$ax^4 = b,\ \ x^4 + x^2 = 20,\ \ \text{or } ax^4 - bx^3 + cx^2 - dx = e.$$

261. A Pure Equation is one which contains only a single power of the unknown quantity; as,

$$ax^5 = b,\ \ x^3 = 8,\ \ \text{or } c^2x^5 = m.$$

PURE QUADRATIC EQUATIONS.

262. A Pure Quadratic Equation is one which contains only the second power of the unknown quantity; as,

$$ax^2 = b,\ \ x^2 = 100,\ \ \text{or } \frac{a^3x^2}{b} = c^4.$$

Note. Pure quadratic equations are sometimes called *incomplete equations* of the second degree.

263. Any numerical pure equation may always be reduced to *two terms,* one containing the unknown quantity, and the other consisting of all the known terms united in one. Thus, the equation

$$\frac{20x^4}{3} - 5x^2 - 4 = \frac{5x^2}{4} + 2\frac{2}{3},$$

Define a Quadratic Equation. A Cubic Equation. A Biquadratic Equation. A Pure Equation. A Pure Quadratic Equation. To how many terms may a pure equation be reduced?

by clearing of fractions and transposing, reduces to

$$5x^2 = 80.$$

In a literal pure equation, all the terms which contain the unknown quantity may be united, since, expressing the same power of the same letter, they are similar; and as the remaining terms are all known quantities, they may be considered as one. Hence the equation

$$ax^2 + bx^2 - cx^2 = d^2 - m^2 + n^2$$

may be thus expressed:

$$(a + b - c)x^2 = (d^2 - m^2 + n^2).$$

Pure equations are therefore sometimes called *binomial equations.*

1. Given $5x^2 = 80$, to find the values of x.

OPERATION.

$$5x^2 = 80 \quad (1)$$
$$x^2 = 16 \quad (2)$$
$$x = \pm 4 \quad (3)$$

By dividing equation (1) by 5 (Art. 151), we obtain (2), and by extracting the square root of both members of the equation (Art. 151), we obtain $x = \pm 4$. As an even root of a positive quantity is either positive or negative (Art. 207), we obtain $x = 4$ and $x = -4$, as the roots of the equation (Art. 155). These we write in one expression, thus, $x = \pm 4$.

Note. It may seem to the student that x should also have the double sign, thus, $\pm x = \pm 4$. The results are the same, however, in either case; for $\pm x = \pm 4$ would include four expressions, $+x = +4$, $+x = -4$, $-x = +4$, and $-x = -4$, of which the third reduces to the second, and the fourth to the first, by a change of signs (Art. 152, Note). Hence we obtain all the possible values much more readily by prefixing the double sign to the second member of the equation.

From the preceding solution it will be seen that

A pure quadratic equation has two roots, which are equal in numerical value, but differ in their signs.

Explain the first operation. Why is not the double sign prefixed to both members? What is said of the number and relation of the roots of a pure quadratic equation?

2. Given $\frac{x}{a}+\frac{b}{x}=\frac{c}{x}-\frac{x}{d}$, to find the values of x.

FIRST OPERATION.

$$\frac{x}{a}+\frac{b}{x}=\frac{c}{x}-\frac{x}{d} \quad (1)$$

$$dx^2+abd=acd-ax^2 \quad (2)$$

$$ax^2+dx^2=acd-abd \quad (3)$$

$$(a+d)x^2=ad(c-b) \quad (4)$$

$$x^2=\frac{ad(c-b)}{a+d} \quad (5)$$

$$x=\pm\sqrt{\frac{ad(c-b)}{a+d}} \quad (6)$$

Clearing equation (1) of fractions gives us (2), transposing gives us (3), and factoring gives us (4). Dividing both members by $a+d$, the coefficient of x^2, gives us (5), the value of x^2; and extracting the square root of both members (Art. 151), gives us (6), the value of x.

SECOND OPERATION.

$$\frac{x}{a}+\frac{b}{x}=\frac{c}{x}-\frac{x}{d} \quad (1)$$

$$a^{-1}x+bx^{-1}=cx^{-1}-d^{-1}x \quad (2)$$

$$a^{-1}x^2+b=c-d^{-1}x^2 \quad (3)$$

$$a^{-1}x^2+d^{-1}x^2=c-b \quad (4)$$

$$(a^{-1}+d^{-1})x^2=c-b \quad (5)$$

$$x^2=\frac{c-b}{a^{-1}+d^{-1}} \quad (6)$$

$$x=\pm\sqrt{\frac{c-b}{a^{-1}+d^{-1}}} \quad (7)$$

$$x=\pm\sqrt{\frac{ad(c-b)}{a+d}} \quad (8)$$

Equation (1) expressed by the aid of negative exponents changes to the form of (2), and the negative exponents of the unknown quantity are removed by multiplying all the terms by x (Art. 153, Note 3), producing equation (3). This equation is solved as in the previous operation, giving (7) as the value of x. This is freed from negative exponents by multiplying both numerator and denominator by ad (Art. 142, Note).

Explain the operations of Example 2.

From the foregoing principles and illustrations we infer that any pure quadratic equation may be solved by the following

RULE.

Obtain the value of the square of the unknown quantity as in simple equations (Art. 159).

Extract the square root of both members of the equation thus obtained.

NOTE 1. A similar application of Ax. 8 will serve to obtain one root of any pure equation of a higher degree. In treating of equations which take the *form* of affected quadratics, we shall have occasion to solve various cubic, biquadratic, and higher *pure* equations.

NOTE 2. It will be observed that many equations, which do not at first appear to be pure quadratics, reduce to such equations after clearing of fractions or performing the operations indicated.

EXAMPLES.

3. Given $3x^2 - 2 = 2x^2 + 2$, to find the values of x.
Ans. $x = \pm 2$.

4. Given $\frac{5x^2}{2} + x^2 = 126$, to find the values of x.
Ans. $x = \pm 6$.

5. Given $y^2 = 9a^6$, to find the values of y.

6. Given $7(2x^2 - 6) + 5(3 - x^2) = 198$, to find x.
Ans. $x = \pm 5$.

7. Given $(x + 1)^2 = 2x + 17$, to find x.
Ans. $x = \pm 4$.

8. Given $x^2 + ab = 5x^2$, to find x.
Ans. $x = \pm \frac{1}{2}\sqrt{ab}$.

9. Given $\frac{2x^2 + 10}{15} = 7 - \frac{50 + x^2}{25}$, to find x.
Ans. $x = \pm 5$.

Repeat the Rule. What is Note 1? Note 2?

10. Given $(x+2)^2 = 4x + 5$, to find x.
Ans. $x = \pm 1$.

11. Given $(2x - 5)^2 = x^2 - 20x + 73$, to find x.

12. Given $4x - 150x^{-1} = x - 3x^{-1}$, to find x.
Ans. $x = \pm 7$.

13. Given $\frac{3}{1+x} + \frac{3}{1-x} = 8$, to find x.
Ans. $x = \pm \frac{1}{2}$.

14. Given $x^2 - \frac{a^2}{4} + 3ab = 2a^2 + b^2$, to find x.
Ans. $x = \pm \left(\frac{3}{2}a - b\right)$.

15. Given $c(x^2 + 4ab + 4bc) = a(a+2c)^2 + dx^2 - a^2 b$, to find x.
Ans. $x = \pm (a+2c)\sqrt{\frac{a-b}{c-d}}$.

16. Given $\frac{x+2}{x-2} + \frac{x-2}{x+2} = \frac{13}{6}$, to find x.
Ans. $x = \pm 10$.

17. Given $x - \frac{3x^2-2}{5x} = 3x^{-1} - \frac{2x^2-5}{3x}$, to find x.
Ans. $x = \pm 2$.

18. Given $y^{-2} - \frac{2}{3y^2+1} = \frac{10}{9(3y^2+1)}$, to find y.
Ans. $y = \pm 3$.

19. Given $\frac{14z^2+16}{21} - \frac{2z^2+8}{8z^2-11} = \frac{2z^2}{3}$, to find z.
Ans. $z = \pm 2$.

20. Given $(a+x)^3 + (a-x)^3 = 2b^3$, to find x.
Ans. $x = \pm \sqrt{\frac{b^3-a^3}{3a}}$.

264. Radical Equations sometimes reduce to the form of pure quadratics. The preliminary reductions should be effected as in Art. 257.

1. Given $22\frac{1}{2} + \sqrt{5(4x^2-1)} = 25$, to find x.
Ans. $x = \pm \frac{3}{4}$.

2. Given $\sqrt{x^2 + \sqrt{x^4 - a^4}} = a$, to find x.
Ans. $x = \pm a$.

3. Given $\sqrt{x^2 - 16} = \frac{3x}{5}$, to find x.

4. Given $\sqrt{a^2 + x^2} = \sqrt[4]{b^4 + x^4}$, to find x.

Ans. $x = \pm\sqrt{\frac{b^4 - a^4}{2a^2}}$.

5. Given $\sqrt{\frac{4x^2 + 128}{3x}} = 2\sqrt{x}$, to find x.

Ans. $x = \pm 4$.

6. Given $\sqrt{x + a} = \sqrt{x + \sqrt{b^2 + x^2}}$, to find x.

Ans. $x = \pm\sqrt{a^2 - b^2}$.

7. Given $x(10 + x^2)^{\frac{1}{2}} = 5 - x^2$, to find x.

Ans. $x = \pm\frac{1}{2}\sqrt{5}$.

8. Given $x + (a^2 + x^2)^{\frac{1}{2}} = 2a^2(a^2 + x^2)^{-\frac{1}{2}}$, to find x.

Ans. $x = \pm\frac{a}{\sqrt{3}}$.

9. Given $\frac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} = b$, to find x.

Ans. $x = \pm\frac{2a\sqrt{b}}{1 + b}$.

Note. Rationalize the denominator of the first member of the last equation, and then extract the square root of both members (Art. 257, Note, and Art. 249, Ex. 7).

10. Given $\sqrt{\frac{x + 3}{x - 3}} + \sqrt{\frac{x - 3}{x + 3}} = 5$, to find x.

Ans. $x = \pm\frac{5}{7}\sqrt{21}$.

Note. Square the last equation as it stands, and thus remove all radicals.

SIMULTANEOUS EQUATIONS.

265. Simultaneous equations (Art. 170) sometimes produce pure equations after elimination. The methods of elimination are the same as in simple equations; but sub-

---

What methods of elimination are here employed?

stitution will be found best adapted to most of the examples which follow.

Some of the equations will be found to be of a higher degree than the second. (Art. 263, Rule, Note 1.)

1. Given $3x^2 - 2y^2 = 40$, and $x - 2y = 0$, to find the values of x and y.

OPERATION.

$$3x^2 - 2y^2 = 40 \qquad (1)$$
$$x = 2y \qquad (2)$$
$$3(2y)^2 - 2y^2 = 40 \qquad (3)$$
$$12y^2 - 2y^2 = 40 \qquad (4)$$
$$10y^2 = 40 \qquad (5)$$
$$y^2 = 4 \qquad (6)$$
$$y = \pm 2 \qquad (7)$$
$$x = 2y = \pm 4 \qquad (8)$$

Equation (2) gives the value of x, in terms of y. Substituting $2y$, this value of x, for x, in equation (1), we obtain (3), which reduces to (6). Extracting the square root of (6), we have (7), the value of y. Substituting the value of y in (2), we have the value of x.

2. Given $\frac{1}{3}x^2 - 3y^2 = 21$, and $\frac{1}{2}x + 2y = 0$, to find x and y. Ans. $x = \pm 12$; $y = \mp 3$.

3. Given $5xy - 3y^2 = 100$, and $5x - 4y = 0$, to find x and y. Ans. $x = \pm 8$; $y = \pm 10$.

4. Given $xy + y^2 = 126$, and $5y = 2x$, to find x and y.

5. Given $4x^2 + 7y^2 = 148$, and $3x^2 - y^2 = 11$, to find x and y. Ans. $x = \pm 3$; $y = \pm 4$.

6. Given $x + y = 3x - 3y$, and $x^3 - y^3 = 56$, to find x and y. Ans. $x = 4$; $y = 2$.

7. Given $x^2 + y^2 : x^2 - y^2 :: 17 : 8$, and $xy^2 = 45$, to find x and y. Ans. $x = 5$; $y = \pm 3$.

8. Given $x^4 + y^4 = 97$, and $9x^2 = 4y^2$, to find x and y. Ans. $x = \pm 2$; $y = \pm 3$.

NOTE. There are also two *imaginary* values for each of the unknown quantities in the last example, viz. $x = \pm 2\sqrt{-1}$, $y = \pm 3\sqrt{-1}$; for $x^2 = 4$ or -4, and $y^2 = 9$ or -9.

Explain the first operation.

PROBLEMS

LEADING TO PURE EQUATIONS.

266. The methods of stating these problems are the same as in the case of those leading to simple equations. (Arts. 160, 167.) Either one or two unknown quantities may be used in many cases.

1. Find two numbers, one of which is three times as great as the other, and the sum of whose squares is 90.

SOLUTION.

Let $x =$ the first number,

and $3x =$ the second number.

Then, $x^2 + 9x^2 = 90$

Uniting terms, $10x^2 = 90$

Dividing by 10, $x^2 = 9$

Evolving, $x = \pm 3$, the first number,

$3x = \pm 9$, the second number.

The only *arithmetical* numbers which will answer the conditions are 3 and 9.

2. Find two numbers, one of which is five times as great as the other, and the difference of whose squares is 96. Ans. 2 and 10.

3. The length of a field is to its breadth as 3 to 2, and its area is 3 acres and 3 roods. What are its dimensions? Ans. Length, 30 rods; breadth, 20 rods.

4. A merchant bought two pieces of cloth, which together measured 36 yards. Each of them cost as many dimes a yard as there were yards in the piece, and their entire prices were as 4 to 1. How many yards were there in each piece?

Ans. 24 yards in one; 12 yards in the other.

Explain the solution of Problem 1.

5. The product of two numbers is 750, and the quotient when one is divided by the other is $3\frac{1}{3}$; find the numbers. Ans. 50 and 15.

6. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but upon the enemy being discovered, the front was increased by 845 men, and by this movement the detachment was drawn up in five equal lines. What was the number of men? Ans. 4550.

7. Find three numbers which shall be to each other as 5, 7, and 9, and the sum of whose squares shall be 620.

Let $5x$, $7x$, and $9x$ represent the numbers.

8. A certain number of boys went out to gather nuts, each taking as many bags as there were boys in all, each bag being of a capacity to contain as many nuts as there were boys. Upon filling the bags, they found them to contain exactly 1000 nuts. How many boys were there? Ans. 10.

9. There are two cubical blocks of stone, one of which contains 117 cubic feet more than the other, and the side of the larger is $2\frac{1}{2}$ times as long as that of the smaller. Required the dimensions of each.

Ans. 5 feet, side of the larger; 2 feet, side of the smaller.

10. Two persons, A and B, set out from different places to meet each other. They started at the same time, and traveled on the direct road between the two places. On meeting, it appeared that A had traveled 18 miles more than B; and that A could have gone B's distance in $15\frac{3}{4}$ days, but B would have been 28 days in going A's distance. How far did each travel?

Ans. A, 72 miles; B, 54 miles.

AFFECTED QUADRATIC EQUATIONS.

267. An AFFECTED QUADRATIC EQUATION is one which contains both the second and first powers of the unknown quantity; as,

$$x^2 + a\,x = b,\quad 2\,x^2 + 16\,x = 40,\quad \text{or } a\,x^2 + b\,x = c.$$

NOTE. Affected quadratic equations are sometimes called *complete equations* of the second degree.

268. Any affected quadratic equation may always be reduced to *three terms*, one containing the second power of the unknown quantity, another its first power, and the remaining one the known terms of the equation. Thus, the equation

$$(x + 1)^2 - \frac{x^2}{3} - x + 2\tfrac{1}{2} = \frac{3\,x}{2} + 8,$$

after performing the operations indicated and clearing of fractions, reduces to

$$4\,x^2 - 3\,x = 27\,;$$

and $a\,x^2 + b\,x - c = b\,x^2 - a\,x + d$

may be thus expressed:

$$(a - b)\,x^2 + (a + b)\,x = (c + d).$$

Affected quadratic equations are therefore sometimes classed among *trinomial equations*.

NOTE. If the first of the three terms is wanting, the equation is evidently of the first degree; if the second is wanting, the equation is a pure quadratic; and if the third is wanting, the equation may be at once reduced to the first degree, by dividing both terms by the unknown quantity.

FIRST METHOD OF COMPLETING THE SQUARE.

269. If the second power of the unknown quantity has any coefficient expressed, the equation may be still

---

Define an Affected Quadratic Equation. To what terms may any affected quadratic equation be reduced? How may it be still further reduced?

further reduced by dividing all its terms by that coefficient. Thus, $3x^2 - 6x = 9$ reduces to $x^2 - 2x = 3$, and $4x^2 - 3x = 27$ reduces to $x^2 - \frac{3}{4}x = \frac{27}{4}$. Hence any affected quadratic equation may be made to assume the form

$$x^2 + px = q,$$

in which p and q are understood to represent any numbers whatever, whether positive or negative, integral or fractional.

1. Given $x^2 + 4x + 4 = 9$, to find the values of x.

OPERATION.

$$\begin{aligned} x^2 + 4x + 4 &= 9 && (1) \\ (x + 2)^2 &= 9 && (2) \\ x + 2 &= \pm 3 && (3) \\ x &= -2 \pm 3 && (4) \\ x &= 1, \text{ or } -5 \end{aligned}$$

VERIFICATION.

$$\begin{aligned} 1^2 + 4 \times 1 + 4 &= 9 && (1) \\ \left.\begin{aligned} (-5)^2 + 4(-5) + 4 &= 9 \\ 25 - 20 + 4 &= 9 \end{aligned}\right\} & && (2) \end{aligned}$$

It is evident that

$$x^2 + 4x + 4$$

is a perfect square of a binomial; for x^2 and 4 are positive squares, while $4x$ is twice the product of their square roots. Equation (1) may therefore take the form of (2). (Art. 90.) This may be regarded as a pure quadratic, in which the unknown quantity is not x, but $x + 2$. Extracting the square root of both members, we have ± 3 as the value of $x + 2$, and the equation is now reduced to a simple one. Taking the upper of the two signs, and transposing 2, we have $x = -2 + 3 = 1$; but taking the lower, we have $x = -2 - 3 = -5$; and these values are found to satisfy the equation.

We thus obtain two roots of the equation, which differ both in sign and in numerical value.

Note. The reason for prefixing the double sign to only the second member of the equation, in extracting the square root, has already been given. (Art. 263, Ex. 1, Note.)

2. Given $x^2 - 6x + 12 = 3$, to find the values of x.

Explain the first operation.

OPERATION.

$$\begin{aligned} x^2-6x+12&=3 &\quad(1)\\ x^2-6x+\ \ 9&=0 &\quad(2)\\ x-3&=\pm 0 &\quad(3)\\ x&=3\pm 0 &\quad(4)\\ x&=3;\ \text{or}\ 3 & \end{aligned}$$

Subtracting 3 from both members of equation (1), we obtain (2), whose first member happens to be a perfect square; for x^2 and 9 are the squares of x and 3, while $6x$ is twice their product. Extracting the square root of each member, we obtain (3), which reduces by transposition to $x = 3$.

It will be seen that the two roots of the above equation are alike, both in sign and in numerical value. Such an equation is said to have *equal roots*.

NOTE. The two roots of a *pure quadratic* equation are alike in *numerical value*, but differ in their *signs* (Art. 263), and hence are not equal, in an algebraic sense. No quadratic equation can have equal roots, unless its second member is 0 when its first member is a perfect square, that is, unless its three terms make a perfect square when collected in one member.

3. Given $x^2 - 8x = 20$, to find the values of x.

OPERATION.

$$\begin{aligned} x^2-8x&=20 &\quad(1)\\ x^2-8x+16&=36 &\quad(2)\\ x-4&=\pm 6 &\quad(3)\\ x&=4\pm 6 &\quad(4)\\ x&=10,\ \text{or}\ -2 & \end{aligned}$$

It is evident that the first member is not a perfect square, as in the first example, neither can it be made such by the transposition of the known term, as in the second example. Such a term must therefore be added to $x^2 - 8x$ as will make it the square of some binomial. As x^2 is the first term of the equation, x, its square root, must be the first term of the binomial sought. The next term of the equation, $8x$, must be twice the product of the two terms of the binomial; and one half of $8x$, or $4x$, must be their product. But $4x$ is the product of 4 and x; hence 4 is the second term of the binomial sought, and its square, or 16, must be added to the first member of the equation to make it a perfect square, and also to the second mem-

---

Explain the second operation. What is said of the roots of the equation? Explain the third operation.

ber to preserve the equality, thus producing equation (2). The square root can now be extracted, thus producing equation (3). Taking 6, the positive root of 36, and transposing and uniting terms, we find $x = 10$; but taking -6, the negative root, we find $x = -2$.

4. Given $1 - \frac{7x}{5} = \frac{5}{3} + \frac{3x^2}{5}$, to find the values of x.

OPERATION.

$$1 - \frac{7x}{5} = \frac{5}{3} + \frac{3x^2}{5} \quad (1)$$

$$5 - 7x = \frac{25}{3} + 3x^2 \quad (2)$$

$$-3x^2 - 7x = \frac{10}{3} \quad (3)$$

$$x^2 + \frac{7}{3}x = -\frac{10}{9} \quad (4)$$

$$x^2 + \frac{7}{3}x + \frac{49}{36} = \frac{9}{36} \quad (5)$$

$$x + \frac{7}{6} = \pm\frac{3}{6} \quad (6)$$

$$x = -\frac{7}{6} \pm \frac{3}{6} \quad (7)$$

$$x = -\frac{4}{6}, \text{ or } -\frac{10}{6}$$

$$x = -\frac{2}{3}, \text{ or } -\frac{5}{3}$$

We first multiply by 5, to free the unknown quantity of fractions, and, after transposing and uniting terms, obtain equation (3). As the square of the unknown quantity must be positive, we then divide all the terms by -3, and obtain (4), the equation in its reduced form.

If the first member of equation (4) is to be made a perfect square, $\frac{7}{3}x$ must be twice the product of the two terms of the root. As x is one of those terms, one half its coefficient, or $\frac{7}{6}$, must be the other term, and the square of $\frac{7}{6}$, or $\frac{49}{36}$, must be added to both members of the equation. Extracting the square root of equation (5), we obtain (6). Taking the positive root of $\frac{9}{36}$, and transposing and uniting terms, we obtain $x = -\frac{4}{6} = -\frac{2}{3}$; but taking the negative root, we obtain $x = -\frac{10}{6} = -\frac{5}{3}$.

It will be seen that the two roots of the above equation have the same sign, but differ in numerical value.

5. Given $x^2 + px = q$, to find the values of x.

Explain the fourth operation.

OPERATION.

$$x^2+px=q \qquad (1)$$

$$x^2+px+\frac{p^2}{4}=q+\frac{p^2}{4} \qquad (2)$$

$$x+\frac{p}{2}=\pm\sqrt{q+\frac{p^2}{4}} \qquad (3)$$

$$x=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}} \qquad (4)$$

As in the other examples, px being twice the product of the two terms of the root of the completed square, and x being one of those terms, $\frac{p}{2}$ must be the other, and x^2+px can be made a perfect square only by adding to it $\frac{p^2}{4}$. After adding the same quantity to the second member of the equation, we extract the square root of both members. The root of the second member, however, can only be expressed. By transposing $\frac{p}{2}$, we find the two values of x to be $-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}$ and $-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}$.

From the foregoing principles and illustrations we infer that any affected quadratic equation may be solved by the following

RULE.

Reduce the equation to three terms, placing the two which contain the unknown quantity on the first side, the higher power first, and the known quantity on the second side. Divide each side by the coefficient of the first term, and the equation will be reduced to the form $x^2+px=q$.

Add the square of half the coefficient of x *to both members of the equation, and the first member will be a complete square.*

Extract the square root of both members, and solve the simple equation thus produced.

NOTE 1. If the coefficient of the square of the unknown quantity happens to be negative, all the signs must be changed. This may be effected by using the *negative* coefficient as a divisor.

Explain the fifth operation. Repeat the Rule. Note 1.

Note 2. After completing the square of the first member of the equation, its first and third terms must be positive squares; but its second term may be either positive or negative. If the second member is then positive and a perfect square, both roots will be real and rational; if it is positive, but not a perfect square, both roots will be real, but irrational (Art. 227); and if it is negative, both roots will be imaginary (Art. 250).

Note 3. The square root of the first member of the equation is composed of the square roots of its first and third terms, connected by the sign of the second term.

The above rule may be applied in the solution of the following

Examples.

6. Given $x^2+2x=8$, to find the values of x.
Ans. $x=2$, or -4.

7. Given $x^2-4x=-4$, to find the values of x.
Ans. $x=2$, or 2.

8. Given $x^2-6x=55$, to find the values of x.

9. Given $x^2+12x+35=0$, to find the values of x.
Ans. $x=-5$, or -7.

10. Given $3z^2+48=30z$, to find the values of z.
Ans. $z=8$, or 2.

11. Given $x^2-2ax=b$, to find the values of x.
Ans. $x=a\pm\sqrt{a^2+b}$.

12. Given $x^2=3x+10$, to find the values of x.
Ans. $x=5$, or -2.

13. Given $2x+60=2x^2$, to find the values of x.
Ans. $x=6$, or -5.

14. Given $4y^2+8y=5$, to find the values of y.
Ans. $y=\frac{1}{2}$, or $-\frac{5}{2}$.

15. Given $5x^2+20=25x$, to find the values of x.
Ans. $x=4$, or 1.

Repeat Note 2. Note 3.

16. Given $3x + 4 = 39x^{-1}$, to find the values of x.
Ans. $x = 3$, or $-4\frac{1}{3}$.

17. Given $5x^2 - 40x = 70$, to find the values of x.
Ans. $x = 4 \pm \sqrt{30}$.

18. Given $3x = 10 + \frac{1}{4}x^2$, to find the values of x.
Ans. $x = 6 \pm 2\sqrt{-1}$.

19. Given $x^2 - 6x = 0$, to find the values of x.
Ans. $x = 6$, or 0.

Note. Such an equation may be solved as an affected quadratic; but one of its roots will be found to be 0, as it evidently should be, since the equation can be at once reduced to a simple one (Art. 268, Note).

20. Given $a^{-1}x + ax^{-1} = 2a^{-1}$, to find the values of x.
Ans. $x = 1 \pm \sqrt{1 - a^2}$.

270. The equation $x^2 + px = q$ may be regarded as the *general* expression of any affected quadratic equation reduced to that form. (Art. 167, Prob. 34.) As this equation has already been solved (Art. 269, Ex. 5), we may use its roots, $-\frac{p}{2} + \sqrt{q + \frac{p^2}{4}}$ and $-\frac{p}{2} - \sqrt{q + \frac{p^2}{4}}$, as the general formulas for the roots of any affected quadratic equation. Instead, then, of going through the full process of solving each equation by itself, according to the foregoing rule, we may write out its roots at once, by substituting the particular values of p and q in the above formulas. Hence,

The roots of any equation reduced to the form $x^2 + px = q$ *may be found by taking one half the coefficient of x, with a contrary sign, plus or minus the square root of the sum of the second member and the square of half the coefficient of x.*

How may any affected quadratic equation be solved without going through the full process of completing the square?

This process is to be employed in the solution of the following

EXAMPLES.

1. Given $x^2 - 8x = 9$, to find the values of x.
   Ans. $x = 4 \pm \sqrt{9 + 16} = 9$, or -1.
2. Given $x^2 + 16x = -55$, to find the values of x.
   Ans. $x = -8 \pm \sqrt{-55 + 64} = -5$, or -11.
3. Given $x^2 - 20x = 800$, to find the values of x.
4. Given $x^2 + 5x = 14$, to find the values of x.
   Ans. $x = -\frac{5}{2} \pm \sqrt{14 + \frac{25}{4}} = 2$, or -7.
5. Given $\frac{x^2}{3} + \frac{3x}{2} = 21$, to find the values of x.
   Ans. $x = 6$, or $-10\frac{1}{2}$.
6. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8$, to find the values of x.
   Ans. $x = \frac{3}{2}$, or $-\frac{5}{6}$.

SECOND METHOD OF COMPLETING THE SQUARE.

271. Any affected quadratic equation whatever may be solved by the method employed in Art. 269. It will be seen, however, that a fraction must be added to complete the square, unless the coefficient of the first power of the unknown quantity in the reduced equation becomes an even whole number; and even then the second member may sometimes be fractional. But by employing another mode of completing the square, sometimes called the "Hindoo method," all fractions can be avoided till the roots are obtained.

272. Any equation may be reduced to three terms, as

Why do we introduce a second method of completing the square?

before, cleared of all fractions, and divided by the greatest common measure of its terms. It will thus be reduced to the form

$$ax^2 + bx = c,$$

in which a, b, and c represent any whole numbers whatever which have no common measure greater than unity.

1. Given $x^2 - 5x + 4 = 0$, to find the values of x.

OPERATION.

$$\begin{aligned} x^2 - 5x + 4 &= 0 && (1) \\ x^2 - 5x &= -4 && (2) \\ 4x^2 - 20x &= -16 && (3) \\ 4x^2 - 20x + 25 &= 9 && (4) \\ 2x - 5 &= \pm 3 && (5) \\ 2x &= 5 \pm 3 && (6) \\ 2x &= 8, \text{ or } 2 && (7) \\ x &= 4, \text{ or } 1 && (8) \end{aligned}$$

VERIFICATION.

$$\begin{aligned} 4^2 - 5 \times 4 + 4 &= 0 && (1) \\ 1^2 - 5 \times 1 + 4 &= 0 && (2) \end{aligned}$$

Transposing the known term to the second member, we have (2). If we wish to complete the square of the first member without introducing fractions, it is evident that the second term should be divisible by 2, as it is twice the product of the two terms of the root. But the first term must be a perfect square; hence, we multiply all the terms of the equation by 4, the smallest even square number, and obtain (3). The square root of the first term is $2x$, which must be the first term of the binomial root, and as $20x$ is twice the product of the two terms of the root, $10x$ must be their product, and the other term must be $\frac{10x}{2x} = 5$. Hence 5^2, or 25, must be added to the first member to render it a perfect square, and to the second member to preserve the equality, thus producing equation (4). Extracting the square root, we obtain (5), which, by transposing and uniting terms, and dividing by 2, the coefficient of x, gives 4 and 1, as the values of x; and these values satisfy the equation.

It will be observed that we have thus avoided the fractions which must be employed in solving this example by the previous rule. (Art. 269, Ex. 15.)

To what form is the equation here reduced? Explain the first operation.

It will also be seen that the quantity added to complete the square is the square of the coefficient of x in equation (2).

2. Given $\frac{18x}{5} - 8x^{-1} = 24 + \frac{2}{5x}$, to find the values of x.

OPERATION.

$$\frac{18x}{5} - 8x^{-1} = 24 + \frac{2}{5x} \qquad (1)$$
$$18x^2 - 40 = 120x + 2 \qquad (2)$$
$$18x^2 - 120x = 42 \qquad (3)$$
$$3x^2 - 20x = 7 \qquad (4)$$
$$9x^2 - 60x = 21 \qquad (5)$$
$$9x^2 - 60x + 100 = 121 \qquad (6)$$
$$3x - 10 = \pm 11 \qquad (7)$$
$$3x = 10 \pm 11 \qquad (8)$$
$$3x = 21, \text{ or } -1 \qquad (9)$$
$$x = 7, \text{ or } -\tfrac{1}{3} \qquad (10)$$

We first remove the denominators and the negative exponent, by multiplying both members by $5x$; then, after transposing and uniting terms, and dividing by 6, the greatest common measure of the three terms of equation (3), we obtain (4) as the reduced equation.

The coefficient of the second term, $-20x$, is an even number; hence there is no necessity for multiplying by 4, as in the last example. But $3x^2$ is not a perfect square, and we must therefore multiply by 3, to render the first term a square, producing equation (5). $3x$, the square root of $9x^2$, must be the first term of the binomial root, and $30x$, one half of $60x$, must be the product of the two terms of the root; hence $\frac{30x}{3x}$, or 10, must be the second term, and its square, 100, must be added to both members. We then extract the square root of both members, and reduce as in the previous example.

The number added to complete the square in the above example is the square of one half the coefficient of x in equation (4).

3. Given $ax^2 + bx = c$, to find the values of x.

Explain the second operation.

OPERATION.

$$a x^2 + b x = c \qquad (1)$$

$$4 a^2 x^2 + 4 a b x = 4 a c \qquad (2)$$

$$4 a^2 x^2 + 4 a b x + b^2 = 4 a c + b^2 \qquad (3)$$

$$2 a x + b = \pm \sqrt{4 a c + b^2} \qquad (4)$$

$$2 a x = -b \pm \sqrt{4 a c + b^2} \qquad (5)$$

$$x = \frac{-b \pm \sqrt{4 a c + b^2}}{2 a} \qquad (6)$$

To make the first term a square, and the second term divisible by 2, we multiply both members by $4 a$, producing equation (2). $2 a x$, the square root of $4 a^2 x^2$, must be the first term of the root, and $\frac{4 a b x}{2}$, or $2 a b x$, must be the product of the two terms; hence $\frac{2 a b x}{2 a x}$, or b, must be the second term of the root, and b^2 must be added to complete the square, producing equation (3). We next extract the square root of the first member, and express the square root of the second; then, by transposing and dividing, we obtain the values of x in equation (6).

The quantity added to complete the square is the square of the coefficient of x in equation (1).

As a, b, and c in the last example may have any value whatever, we derive from the solution of that equation the following

RULE.

Reduce the equation to three integral terms, placing the two which contain the unknown quantity on the first side, the higher power first, and the known quantity on the second side. Divide the three terms by their greatest common measure, and the equation will be reduced to the form $a x^2 + b x = c$.

Multiply both members of the equation by four times the coefficient of x^2, *and add to each the square of the coefficient of* x.

Extract the square root of both members, and solve the simple equation thus produced.

Explain the third operation. Repeat the Rule.

Note 1. It must be observed, that in this rule we take the square of the coefficient which x has *before* it is multiplied; but in the previous one we take the square of one half the coefficient which it has *after* it is divided.

Note 2. Any quadratic equation may also be solved by using the *coefficient* of x^2 as a multiplier, instead of four times that coefficient, and adding the square of *one half the coefficient* of x, instead of the square of that coefficient. If the coefficient of x is an *even* number, this method will avoid fractions, and at the same time make each term only one fourth as great as it would be by the rule given above.

Note 3. If the coefficient of x^2 is negative in the reduced form of the equation, all the signs must be changed. This may be effected by including the negative sign in the multiplier.

Note 4. The formula $x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}$, obtained by the solution of Example 3, may be used for the solution of any quadratic equation of the form $ax^2 + bx = c$, in the same manner as the formula $x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}$ is used in Art. 270.

The use of fractions is to be avoided in the solution of the following

Examples.

4. Given $x^2 - 7x + 6 = 0$, to find the values of x.
Ans. $x = 6$, or 1.

5. Given $x^2 + \frac{x}{2} = 3$, to find the values of x.
Ans. $x = 1\frac{1}{2}$, or -2.

6. Given $10x = 6x^2 + 4$, to find the values of x.

7. Given $5x^2 = 57 - 4x$, to find the values of x.
Ans. $x = 3$, or $-3\frac{4}{5}$.

8. Given $\frac{a^2 - b^2}{c} = 2ax - cx^2$, to find the values of x.
Ans. $x = \frac{a \pm b}{c}$.

9. Given $\frac{x}{a} + bx^{-1} = c$, to find the values of x.
Ans. $x = \frac{ac \pm \sqrt{a^2c^3 - 4ab}}{2}$.

Repeat Note 1. Note 2. Note 3. Note 4.

10. Given $(x+1)\,(2x+3)=4x^2-22$, to find the values of x. Ans. $x=5$, or $-\frac{5}{2}$.

11. Given $\frac{3}{4}(y^2-3)=\frac{1}{8}(y-3)$, to find the values of y. Ans. $y=\frac{5}{3}$, or $-\frac{3}{2}$.

12. Given $x^{-2}+\frac{5}{2}x^{-1}+\frac{3}{2}=0$, to find the values of x. Ans. $x=-1$, or $-\frac{2}{3}$.

Note. When *all* the exponents of the unknown quantity are negative, as in the last equation, the negative exponents may be retained until the value of x^{-1} is found. The value of x will be its reciprocal. If it is preferred, however, the negative exponents may be removed at once, as in previous examples.

THIRD METHOD OF COMPLETING THE SQUARE.

273. The following statement includes all the methods of completing the square already given, for it is founded directly upon the nature of the square of a binomial. It will be seen that it is substantially the method employed in the solutions on which the rules have been founded, the main difference being that we here multiply the divisor by two, instead of dividing the dividend by that number.

The terms of the equation being arranged in the same manner as before,

Make the coefficient of the first term a positive square, either by multiplication or by division. Divide the second term by twice the square root of the first, and add the square of the quotient to both sides.

The character of the solution will depend upon the multiplier or divisor used to render the coefficient of the first term a perfect square. If the smallest factor or divisor be used, this method will, of course, frequently

---

What statement will include all the rules for completing the square? On what does the character of the solution depend?

require the use of fractions; but it may occasionally be applied to advantage, and give a solution preferable to that obtained by either of the rules before given, especially when the coefficient of the first term is either a square, or contains a square factor, as in the following

EXAMPLES.

1. Given $9x^2 - 6x = 8$, to find the values of x.
Ans. $x = \frac{4}{3}$, or $-\frac{2}{3}$.

NOTE. $\sqrt{9x^2} = 3x$, and $\frac{6x}{6x} = 1$. Hence the addition of 1 will complete the square.

2. Given $4x^2 + 4x = 3$, to find the values of x.

3. Given $27ax^2 + 6bx = \frac{b^2}{a}$, to find the values of x.
Ans. $x = \frac{b}{9a}$, or $-\frac{b}{3a}$.

NOTE. Multiply by $3a$, or multiply by a and divide by 3. The former course will avoid fractions, and $\left(\frac{18abx}{18ax}\right)^2 = b^2$ will complete the square.

4. Given $50x^2 + 4x = \frac{9}{10}$, to find the values of x.
Ans. $x = \frac{1}{10}$, or $-\frac{9}{50}$.

NOTE. Either divide by 2, or multiply by 2. Fractions must be used in either case.

5. Given $5x^{-1}(5x^{-1} - 12) = -36$, to find the values of x.
Ans. $x = \frac{5}{6}$.

6. Given $\frac{x^2}{16} + 4x = 17$, to find the values of x.
Ans. $x = 4$, or -68.

7. Given $\frac{x^2}{12} + 3x = 21$, to find the values of x.
Ans. $x = 6$, or -42.

When may this method be used in preference to the rules before given?

274. The following equations are to be solved by either of the methods which have been explained, care being taken to select the method best adapted to the example under consideration.

The radical equations introduced in this connection are such as reduce to quadratics by the methods already explained and applied. (Art. 257.)

Examples.

1. Given $x^2 - 14x = 120$. Ans. $x = 20$, or -6.

2. Given $x^2 - \frac{3x}{2} = 27$. Ans. $x = 6$, or $-4\frac{1}{2}$.

3. Given $2x^2 - 10x = 100$.

4. Given $16x^{-1} - 4 = 12x^{-2}$. Ans. $x = 3$, or 1.

5. Given $x^2 - \frac{3}{10}x = \frac{1}{10}$. Ans. $x = \frac{1}{2}$, or $-\frac{1}{5}$.

6. Given $\frac{2x^2}{3} + 3\frac{1}{2} = \frac{x}{2} + 8$.

7. Given $\frac{z^2}{12} + \frac{20}{3} = 2z$. Ans. $z = 20$, or 4.

8. Given $2x^2 + 15 = 3x$. Ans. $x = \frac{3 \pm \sqrt{-111}}{4}$.

9. Given $x^2 - 6x + 19 = 13$, to find the approximate values of x. Ans. $x = 4.732$, or 1.268.

10. Given $4ax^2 - 2bx = c$.
Ans. $x = \frac{b \pm \sqrt{4ac + b^2}}{4a}$.

11. Given $x^2 - 4 = 16 - (x - 2)^2$.
Ans. $x = 4$, or -2.

12. Given $(3x - 5)(2x - 5) = (x + 3)(x - 1)$.
Ans. $x = 4$, or $\frac{7}{5}$.

13. Given $(2x - 3)^2 = 8x$. Ans. $x = \frac{9}{2}$, or $\frac{1}{2}$.

14. Given $\frac{1}{6}(x - 3)^2 + \frac{9}{2} = x$. Ans. $x = 6$, or 6.

15. Given $x^2 + (x + 1)^2 = \frac{13}{6}x(x + 1)$.
Ans. $x = 2$, or -3.

How are the equations of Art. 274 to be solved?

16. Given $3(2-x)+2(3-x)=2(4+3x^2)$.

Ans. $x=\frac{1}{2}$, or $-\frac{4}{3}$.

17. Given $4(x-1)-\frac{x-1}{2x}=3\frac{3}{4}$.

Ans. $x=2$, or $\frac{1}{16}$.

18. Given $\frac{6}{x+1}+\frac{2}{x}=3$. Ans. $x=2$, or $-\frac{1}{3}$.

19. Given $\frac{7}{x+1}+\frac{2}{x}-5=0$, to find the approximate values of x. Ans. $x=1.148$, or -0.348.

20. Given $\frac{x}{x+60}=\frac{7}{3x-5}$. Ans. $x=14$, or -10.

21. Given $8x+11+7x^{-1}=3+\frac{65x}{7}$.

Ans. $x=7$, or $-\frac{7}{9}$.

22. Given $\frac{21}{5-x}-\frac{x}{7}=3\frac{2}{7}$. Ans. $x=-2$, or -16.

23. Given $\frac{x^2-5x}{x+3}=x-3+x^{-1}$.

Ans. $x=1$, or $\frac{3}{5}$.

24. Given $\frac{1}{x+6}+8x^{-1}=\frac{3}{x+2}$.

Ans. $x=-4$, or -4.

25. Given $\frac{x+3}{x+2}+\frac{x-3}{x-2}=\frac{2x-3}{x-1}$.

Ans. $x=4$, or 0.

Note. If the second member of the reduced equation becomes 0 *before* completing the square, one of the roots will be 0 (Ex. 19, Note, Art. 269); but if the second member becomes 0 *after* completing the square, the roots will be equal (Ex. 2 and Note, Art. 269).

26. Given $\frac{3x-2}{2x-5}+\frac{2x-5}{3x-2}=\frac{10}{3}$.

Ans. $x=\frac{13}{3}$, or $\frac{1}{7}$.

27. Given $x^2+ax+bx+ab=0$.

Ans. $x=-a$, or $-b$.

28. Given $adx-acx^2=bcx-bd$.

Ans. $x=\frac{d}{c}$, or $-\frac{b}{a}$.

X

29. Given $(a+b)x^2 - cx = \frac{ac}{a+b}$.

Ans. $x = \frac{c \pm \sqrt{c^2 + 4ac}}{2(a+b)}$.

30. Given $\sqrt{x^5} + \sqrt{x^3} = 6\sqrt{x}$. Ans. $x = 2$, or -3.

31. Given $(4x+5)^{\frac{1}{2}}(7x+1)^{\frac{1}{2}} = 30$.

Ans. $x = 5$, or $-\frac{179}{28}$.

32. Given $\sqrt{2x} - 7x = -52$. Ans. $x = 8$, or $\frac{338}{49}$.

33. Given $\sqrt{x+3} + \sqrt{x+8} = 5\sqrt{x}$.

Ans. $x = 1$, or $\frac{1}{21}$.

34. Given $\sqrt{x+4} - \sqrt{x} = \sqrt{x + \frac{3}{2}}$.

Ans. $x = \frac{1}{2}$, or $-\frac{25}{6}$.

35. Given $\frac{x^3+1}{x^3-1} = x + \left(\frac{6}{x}\right)^{\frac{1}{2}}$. Ans. $x = \frac{3}{2}$, or $\frac{2}{3}$.

36. Given $\sqrt{x} + \sqrt{a-x} = \sqrt{b}$.

Ans. $x = \frac{a \pm \sqrt{2ab - b^2}}{2}$.

EQUATIONS IN THE QUADRATIC FORM.

275. The rules already given for the solution of quadratic equations will apply to any equation which can be made to assume the quadratic form.

An equation takes the quadratic form when it is expressed in three terms, and of the two terms which contain the unknown quantity, *one has an exponent twice as great as the other.* The quadratic form, then, is

$$ax^{2n} + bx^n = c, \text{ or } x^{2n} + px^n = q,$$

in which n may have any value whatever, positive or negative, integral or fractional. x may also represent

---

To what other equations may the rules for quadratics be extended? What is the quadratic form?

either the unknown quantity itself, or some expression containing the unknown quantity.

276. Higher Equations in the quadratic form usually reduce to pure equations of some higher degree than the first, after the completion of the square and extraction of the square root. The solution must, therefore, be completed by the rule for pure equations. (Art. 263, Rule, Note 1.)

1. Given $x^6 + 3x^3 = 810$, to find the values of x.

OPERATION.

$$x^6 + 3x^3 = 810 \tag{1}$$
$$4x^6 + 12x^3 = 3240 \tag{2}$$
$$4x^6 + 12x^3 + 9 = 3249 \tag{3}$$
$$2x^3 + 3 = \pm 57 \tag{4}$$
$$2x^3 = -3 \pm 57 \tag{5}$$
$$2x^3 = 54, \text{ or } -60 \tag{6}$$
$$x^3 = 27, \text{ or } -30 \tag{7}$$
$$x = 3, \text{ or } \sqrt[3]{-30} \tag{8}$$

VERIFICATION.

$$729 + 3 \times 27 = 810 \tag{1}$$
$$900 + 3(-30) = 810 \tag{2}$$

This equation evidently has the quadratic form, since x^6 is the square of x^3. As the coefficient of the second term is an odd number, we avoid fractions by using the second method of completing the square; that is, we multiply by 4, and add the square of 3 to both members. Extracting the square root of (3), we obtain (4), a pure cubic equation, which reduces to (7) by transposing, uniting, and dividing. Extracting the cube root of both members, we obtain (8).

2. Given $x^2 + 4x^{-2} = 5$, to find the values of x.

FIRST OPERATION.

$$x^2 + \frac{4}{x^2} = 5 \tag{1}$$
$$x^4 + 4 = 5x^2 \tag{2}$$
$$x^4 - 5x^2 = -4 \tag{3}$$
$$x^4 - 5x^2 + \tfrac{25}{4} = \tfrac{9}{4} \tag{4}$$
$$x^2 - \tfrac{5}{2} = \pm \tfrac{3}{2} \tag{5}$$
$$x^2 = 4, \text{ or } 1 \tag{6}$$
$$x = \pm 2, \text{ or } \pm 1 \tag{7}$$

We first remove the denominator by multiplying both members of the equation by x^2, and then transpose the terms, producing equation (3), which is evidently in the quadratic form. Completing the square by the first method, extracting the square root, transposing and uniting terms, we obtain the value of x^2 in equation (6). Extracting the square root again, we have the value of x in equation (7).

SECOND OPERATION.

$$x^2 + 4x^{-2} = 5 \qquad (1)$$
$$x^2 + 4 + 4x^{-2} = 9 \qquad (2)$$
$$x + 2x^{-1} = \pm 3 \qquad (3)$$
$$x^2 + 2 = \pm 3x \qquad (4)$$
$$x^2 \mp 3x = -2 \qquad (5)$$
$$x^2 \mp 3x + \tfrac{9}{4} = \tfrac{1}{4} \qquad (6)$$
$$x \mp \tfrac{3}{2} = \pm \tfrac{1}{2} \qquad (7)$$
$$x = \pm \tfrac{3}{2} \pm \tfrac{1}{2} \qquad (8)$$
$$x = \pm 2, \text{ or } \pm 1 \qquad (9)$$

As x^2 and $4x^{-2}$ are both positive squares, and the letters cancel each other when those terms or their roots are multiplied together, we may complete the square by supplying the middle term, which must be twice the product of the square roots of x^2 and $4x^{-2}$, or 4. Extracting the square root, we have (3). Multiplying by x to remove the negative exponent (Art. 153, Note 3), we find that the equation becomes an affected quadratic, instead of a pure quadratic. Solving equation (5), we obtain the same results as by the other process.

3. Given $x^4 - 9x^2 + 20 = 0$, to find the values of x.
Ans. $x = \pm\sqrt{5}$, or ± 2.

4. Given $x^6 - 35x^3 + 216 = 0$, to find the values of x.
Ans. $x = 3$, or 2.

5. Given $5x^6 - 90x^3 - 270 = 945$, to find the values of x.
Ans. $x = 3$, or $\sqrt[3]{-9}$.

6. Given $x^{10} + 31x^5 = 32$, to find the values of x.
Ans. $x = 1$, or -2.

7. Given $x^{2n} - 4x^n = 10$, to find the values of x.
Ans. $x = (2 \pm \sqrt{14})^{\frac{1}{n}}$.

8. Given $x^2 + 1225x^{-2} = 74$, to find the values of x.
Ans. $x = \pm 7$, or ± 5.

277. Radical Equations sometimes take the quadratic form, and reduce to pure equations.

Explain the first and second operations of Example 2.

e

Note. Some of the following equations may be changed to true quadratics by removing the radicals, as has heretofore been done (Art. 274). It is intended, however, that they should be solved by the *quadratic form*, and not as true quadratics.

1. Given $x + 2\sqrt{x} = 15$, to find the values of x.

OPERATION.

$$\begin{aligned} x + 2\sqrt{x} &= 15 &\quad (1)\\ x + 2\sqrt{x} + 1 &= 16 &\quad (2)\\ \sqrt{x} + 1 &= \pm 4 &\quad (3)\\ \sqrt{x} &= 3, \text{ or } -5 &\quad (4)\\ x &= 9, \text{ or } 25 &\quad (5) \end{aligned}$$

VERIFICATION.

$$\begin{aligned} 9 + 2 \times 3 &= 15 &\quad (1)\\ 25 + 2(-5) &= 15 &\quad (2) \end{aligned}$$

The exponent of the first term is 1, or $\frac{2}{2}$, and that of the second is $\frac{1}{2}$, for $2\sqrt{x} = 2x^{\frac{1}{2}}$; hence the equation has the quadratic form. Completing the square by the first method, and extracting the square root, we obtain (3); transposing and uniting, we have (4); and squaring both members, we have (5).

In verifying these values, we find that 9 is limited to the positive square root, while 25 is limited to the negative square root, as those roots only will satisfy the equation. It will be seen that $(+3)^2$ and $(-5)^2$ are, then, the real roots of the equation, as we might infer from the origin of 9 and 25, equation (4).

2. Given $3x^2 + x^{\frac{7}{6}} = 3104x^{\frac{1}{3}}$, to find the values of x.

OPERATION.

$$\begin{aligned} 3x^2 + x^{\frac{7}{6}} &= 3104x^{\frac{1}{3}} &\quad (1)\\ 3x^{\frac{5}{3}} + x^{\frac{5}{6}} &= 3104 &\quad (2)\\ 36x^{\frac{5}{3}} + 12x^{\frac{5}{6}} &= 37248 &\quad (3)\\ 36x^{\frac{5}{3}} + 12x^{\frac{5}{6}} + 1 &= 37249 &\quad (4)\\ 6x^{\frac{5}{6}} + 1 &= \pm 193 &\quad (5)\\ 6x^{\frac{5}{6}} &= 192, \text{ or } -194 &\quad (6)\\ x^{\frac{5}{6}} &= 32, \text{ or } -\tfrac{97}{3} &\quad (7)\\ x^{\frac{1}{6}} &= 2, \text{ or } (-\tfrac{97}{3})^{\frac{1}{5}} &\quad (8)\\ x &= 64, \text{ or } (-\tfrac{97}{3})^{\frac{6}{5}} &\quad (9) \end{aligned}$$

Dividing both members of the equation by $x^{\frac{1}{3}}$, we obtain (2), which is in the quadratic form, because the exponent $\frac{5}{3}$ is twice as great as the exponent $\frac{5}{6}$, and $x^{\frac{5}{3}}$ is therefore the square of $x^{\frac{5}{6}}$. Multiplying by 4 × 3, or 12, adding 1, the square of the coefficient of $x^{\frac{5}{6}}$, extracting the square root, transposing, uniting, and dividing by 6, we obtain the value of $x^{\frac{5}{6}}$

Explain the operation of Example 1. Of Example 2.

in equation (7). Extracting the fifth root of both sides of the equation, we obtain (8); and raising both sides to the sixth power, we obtain the values of x in equation (9).

As we extract the corresponding root to remove the numerator of the exponent of x, and raise to a corresponding power to remove its denominator, the effect is the same as if we at once *transfer the exponent of x to the second member by inverting it.*

Note. It may be well to state in connection the three ways in which a quantity may be transferred from one member of an equation to the other, corresponding with the three changes of addition or subtraction (Art. 38, Ax. 1, 2), multiplication or division (Ax. 3, 4), and involution or evolution (Ax. 8).

1. Any *term* may be transposed from one member of an equation to the other by changing its sign, that is, *the sign of its coefficient.*

2. A *factor* of either *member* of an equation may be transposed to the other member by changing *the sign of its exponent.*

3. An *exponent* of either *member* of an equation may be transferred to the other member by *inverting it.*

It will be seen that the factor and exponent must belong to the *whole member*, not to a single term only, nor, in the case of the exponent, to a single factor only.

3. Given $x + 4\sqrt{x} = 21$, to find the values of x.
Ans. $x = 9$, or 49.

4. Given $x^{-1} + x^{-\frac{1}{2}} = 6$, to find the values of x.
Ans. $x = \frac{1}{4}$, or $\frac{1}{9}$.

5. Given $x^{\frac{4}{3}} + 10\,x^{\frac{2}{3}} = 171$, to find the values of x.
Ans. $x = 27$, or $(-19)^{\frac{3}{2}}$.

6. Given $5\,y^{\frac{1}{2}} + y^{\frac{1}{4}} = 22$, to find the values of y.
Ans. $y = 16$, or $\left(-\frac{11}{5}\right)^4$.

7. Given $\sqrt[5]{x} + \sqrt[5]{x^2} = 6$, to find the values of x.
Ans. $x = 32$, or -243.

How may an exponent be transferred from one member of an equation to the other?

8. Given $x^{\frac{1}{n}} - x^{\frac{2}{n}} + 2 = 0$, to find the values of x.
Ans. $x = 2^n$, or $(-1)^n$.

9. Given $x^n + p\,x^{\frac{n}{2}} = q$, to find the values of x.
Ans. $x = (-\frac{1}{2}p \pm \sqrt{q + \frac{1}{4}p^2})^{\frac{2}{n}}$.

10. Given $\sqrt{x^5} - 3\,x = 40\,x^{-\frac{1}{2}}$, to find the values of x.
Ans. $x = 4$, or $(-5)^{\frac{2}{3}}$.

278. Polynomials may sometimes take the place of the unknown quantity, as the basis of the quadratic form. These polynomials may have the exponents 2 and 1, or they may have higher or fractional exponents, bearing the same ratio.

Note. Most of the equations which belong to this class must also be considered either *higher* or *radical* equations. The first Note found in the last Article will apply to the latter.

1. Given $x - \sqrt{x + 5} = 1$, to find the values of x.

OPERATION.

$$x - \sqrt{x+5} = 1 \qquad (1)$$
$$x + 5 - \sqrt{x+5} = 6 \qquad (2)$$
$$(x+5) - (x+5)^{\frac{1}{2}} = 6 \qquad (3)$$
$$(x+5) - (x+5)^{\frac{1}{2}} + \tfrac{1}{4} = \tfrac{25}{4} \qquad (4)$$
$$(x+5)^{\frac{1}{2}} - \tfrac{1}{2} = \pm\tfrac{5}{2} \qquad (5)$$
$$(x+5)^{\frac{1}{2}} = 3, \text{ or } -2 \qquad (6)$$
$$x + 5 = 9, \text{ or } 4 \qquad (7)$$
$$x = 4, \text{ or } -1 \qquad (8)$$

VERIFICATION.

$$4 - 3 = 1 \qquad (1)$$
$$-1 - (-2) = 1 \qquad (2)$$

We first add 5 to both members of the equation, in order that we may make the quantity without the radical the same as that within. The equation then assumes the quadratic form, $(x + 5)$

being its basis, instead of x. The coefficient of $(x+5)^{\frac{1}{2}}$ is 1, and we therefore add $(\frac{1}{2})^2$ to both members to complete the square. Extracting the square root, transposing, and uniting, we find the value of $(x+5)^{\frac{1}{2}}$ in equation (6). Squaring, and transposing 5, we have the value of x.

In verifying these values of x, we are obliged to take the *positive* root of $x+5$ when $x=4$, but the *negative* root when $x=-1$, as these only will satisfy the equation.

2. Given $(x-5)^3-3(x-5)^{\frac{3}{2}}=40$, to find the values of x.

OPERATION.

$$(x-5)^3-3(x-5)^{\frac{3}{2}}=40 \qquad (1)$$

$$y^2=(x-5)^3, \text{ and } y=(x-5)^{\frac{3}{2}} \qquad (2)$$

$$y^2-3y=40 \qquad (3)$$

$$y=8, \text{ or } -5 \qquad (4)$$

$$(x-5)^{\frac{3}{2}}=8, \text{ or } -5 \qquad (5)$$

$$x-5=4, \text{ or } (-5)^{\frac{2}{3}} \qquad (6)$$

$$x=9, \text{ or } 5+(-5)^{\frac{2}{3}} \qquad (7)$$

$$x=9, \text{ or } 5+\sqrt[3]{25} \qquad (8)$$

This equation is of the quadratic form, because the exponent 3 is twice as great as $\frac{3}{2}$. We may carry through the solution without any change of letters, as in the last example; or we may substitute y^2 for $(x-5)^3$, and y for $(x-5)^{\frac{3}{2}}$, when equation (1) becomes (3). This last equation, solved by either of the rules for quadratics, gives (4), or, replacing the value of y, (5), which readily reduces to (7) or (8).

3. Given $(x-1)^2-x=-\frac{1}{4}$, to find the values of x.

Ans. $x=2\frac{1}{2}$, or $\frac{1}{2}$.

NOTE. The above equation, by adding 1 to both members, assumes the form $(x-1)^2-(x-1)=\frac{3}{4}$, and may thus be solved. It will be seen, however, that the given equation can be readily reduced to a common quadratic by expanding $(x-1)^2$ and uniting terms, as in the Examples of Art. 274.

Explain the first operation. The second.

4. Given $(y^2-4y)^2-6(y^2-4y)+5=0$, to find the values of y. Ans. $y=5$, or -1, or $2\pm\sqrt{5}$.

5. Given $x^2-2x+6\sqrt{x^2-2x+5}=11$, to find the values of x. Ans. $x=1$, or $1\pm2\sqrt{15}$.

6. Given $\sqrt{x+2}+2\sqrt[4]{x+2}=8$, to find the values of x. Ans. $x=14$, or 254.

7. Given $(x^2+7)^{\frac{6}{5}}+2(x^2+7)^{\frac{3}{5}}=80$, to find the real values of x. Ans. $x=\pm5$.

8. Given $(x-3)^4+4(x-3)^2=117$, to find the values of x. Ans. $x=6$, or 0, or $3\pm\sqrt{-13}$.

9. Given $\sqrt{z+12}+\sqrt[4]{z+12}=6$, to find the values of z. Ans. $z=4$, or 69.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

279. The DEGREE of an equation containing more than one unknown quantity is indicated by the highest sum of the exponents of the unknown quantities contained in any one of its terms. (Art. 145.) Thus,

$5xy+2x+3y=43$ is of the second degree,

and $ax^2y+b^2xy=c^4$ is of the third degree.

280. A HOMOGENEOUS EQUATION is one whose terms, except those which contain only known quantities, are homogeneous with respect to the unknown quantities. (Art. 30.) Thus, the equations

$$5xy+2x^2+3y^2=65, \text{ and } ax^2y+bxy^2=c^4,$$

are homogeneous, for in each equation the sum of the exponents of the unknown quantities is the same in every term which contains an unknown quantity.

How is the degree of an equation containing more than one unknown quantity indicated? Define a Homogeneous Equation.

281. A SYMMETRICAL EQUATION is one in which the unknown quantities are similarly involved. Thus, the equations

$$3x^2y^2=108,\ \frac{x}{y}+\frac{y}{x}=\frac{13}{6},\ \text{and}\ x^2+y^2+xy-2x-2y=9,$$

are symmetrical; for in each of the equations x and y are affected by the same coefficients and exponents, and perform the same office.

NOTE. **Many equations are both homogeneous and symmetrical; as**

$$3x^2+3y^2=39,\ \text{or}\ x^2+xy+y^2=b^2.$$

282. In general, two quadratic equations containing two unknown quantities will produce an equation of the fourth degree after elimination. The rules for quadratics are not, therefore, sufficient to solve *all* simultaneous equations of the second degree. Most of those which are capable of solution by means of rules already given may be included in three cases:—

I. When one equation is of the first degree and the other of the second.

II. When both equations are homogeneous and of the second degree.

III. When the equations are symmetrical.

CASE I.

283. When one equation is of the first degree and the other of the second.

Equations belonging to this class can always be solved. It is usually most convenient to find an expression for

---

Define a Symmetrical Equation. Are the rules for quadratics sufficient to solve all simultaneous equations of the second degree? What ones can be solved? How are equations belonging to Case I. usually solved?

the value of one of the unknown quantities in the simple equation, and eliminate by substitution. Examples have already been given in which a pure equation is thus obtained. (Art. 265.)

EXAMPLES.

1. Given $5(x^2-x)+3xy-2y^2=10$, and $2x+y=7$, to find the values of x and y.

OPERATION.

$$5(x^2-x)+3xy-2y^2=10 \qquad (1)$$

$$2x+y=7 \qquad (2)$$

From (2), $y=7-2x$ (3)

Subs. in (1), $5(x^2-x)+3x(7-2x)-2(7-2x)^2=10$ (4)

Expanding, $5x^2-5x+21x-6x^2-98+56x-8x^2=10$ (5)

Uniting terms, $-9x^2+72x=108$ (6)

Dividing by -9, $x^2-8x=-12$ (7)

Completing square, $x^2-8x+16=4$ (8)

Evolving, $x-4=\pm 2$ (9)

Whence, $x=6$, or 2

Substituting in (3), $y=7-12$, or $7-4$

Whence, $y=-5$, or 3

VERIFICATION.

$$\text{First set of values,}\ \left\{\begin{array}{rr}150-90-50=10 & (1)\\ 12-5=7 & (2)\end{array}\right.$$

$$\text{Second set of values,}\ \left\{\begin{array}{rr}10+18-18=10 & (1)\\ 4+3=7 & (2)\end{array}\right.$$

It will be observed that the values of x and y must be taken in the same order; that is, when $x=6$, $y=-5$; and when $x=2$, $y=3$.

Explain the operation.

2. Given $x+y=7$, and $x^2+2y^2=34$, to find the values of x and y.

$$\text{Ans.}\ \begin{cases} x=4, \text{ or } \frac{16}{3}. \\ y=3, \text{ or } \frac{5}{3}. \end{cases}$$

3. Given $x-\frac{x-y}{2}=4$, and $y-\frac{x+3y}{x+2}=1$, to find the values of x and y.

$$\text{Ans.}\ \begin{cases} x=2, \text{ or } 5. \\ y=6, \text{ or } 3. \end{cases}$$

4. Given $x+4y=23$, and $x^2+3xy=54$, to find the values of x and y.

$$\text{Ans.}\ \begin{cases} x=3, \text{ or } -72. \\ y=5, \text{ or } \frac{95}{4}. \end{cases}$$

5. Given $49x^2=36y^2$, and $x(2x+\frac{1}{2})+3xy-y(6y+5)+128=0$, to find the values of x and y.

$$\text{Ans.}\ \begin{cases} x=6, \text{ or } -8. \\ y=7, \text{ or } -\frac{28}{3}. \end{cases}$$

Note. It is evident that one of the equations can be readily reduced to a simple one.

CASE II.

284. When both equations are homogeneous and of the second degree.

Equations belonging to this class can always be solved. It is usually most convenient to substitute for one of the unknown quantities the product of the other by a third unknown quantity.

Examples.

1. Given $2y^2-4xy+3x^2=17$, and $y^2-x^2=16$, to find the values of x and y.

How are equations belonging to Case II. usually solved?

OPERATION.

$2y^2 - 4xy + 3x^2 = 17$ (1)

$y^2 - x^2 = 16$ (2)

Let $y = vx$ (3)

Subs. in (1), $2v^2x^2 - 4vx^2 + 3x^2 = 17$ (4)

Subs. in (2), $v^2x^2 - x^2 = 16$ (5)

From (4), $x^2 = \frac{17}{2v^2 - 4v + 3}$ (6)

From (5), $x^2 = \frac{16}{v^2 - 1}$ (7)

Hence, $\frac{17}{2v^2 - 4v + 3} = \frac{16}{v^2 - 1}$ (8)

Clearing of fractions, $17v^2 - 17 = 32v^2 - 64v + 48$ (9)

Tr. and uniting, $-15v^2 + 64v = 65$ (10)

Dividing by —15, $v^2 - \frac{64}{15}v = -\frac{65}{15}$ (11)

Whence, $v = \frac{13}{5}$, or $\frac{5}{3}$

Substituting in (7), $x^2 = \frac{16}{\frac{169}{25} - 1}$, or $\frac{16}{\frac{25}{9} - 1}$

Reducing, $x^2 = \frac{25}{9}$, or 9

Evolving, $x = \pm\frac{5}{3}$, or ± 3

Substituting in (3), $y = \pm\frac{5}{3} \times \frac{13}{5}$, or $\pm 3 \times \frac{5}{3}$

Reducing, $y = \pm\frac{13}{3}$, or ± 5

2. Given $y^2 - x^2 = 3$, and $y^2 - 2xy + 2x^2 = 2$, to find the values of x and y.

Ans. $\begin{cases} x = \pm 1, \text{ or } \pm\frac{1}{5}\sqrt{5}. \\ y = \pm 2, \text{ or } \pm\frac{4}{5}\sqrt{5}. \end{cases}$

3. Given $x^2 + 3xy - y^2 = 27$, and $3x^2 + 2xy = 63$, to find the values of x and y.

Ans. $\begin{cases} x = \pm 3, \text{ or } \pm\frac{21}{23}\sqrt{23}. \\ y = \pm 6, \text{ or } \pm\frac{3}{23}\sqrt{23}. \end{cases}$

Note. If either x or y be directly eliminated from such equations as the above, the result will be a biquadratic equation in the quadratic form. (Art. 275.) Two homogeneous quadratic equations, containing two unknown quantities, may therefore be solved in that manner, without the aid of a third unknown quantity.

Explain the operation. What method of solving homogeneous equations is mentioned in the Note?

It will be seen that the square root must be taken twice, whatever the method used, and that each unknown quantity must have four values, two of which differ only in their signs. (Art. 263.)

CASE III.

285. When the equations are symmetrical.

It is often convenient to combine and simplify quadratic and higher equations belonging to this class, before attempting to eliminate either unknown quantity. The proper application of the various expedients employed must be learned by experience, as the details vary with each new class of examples. The student must be thoroughly conversant with the forms of the powers of binomials, the principles of factoring, and especially with the relations existing between the sum, difference, and product of two quantities.

EXAMPLES.

1. Given $x + y = 7$, and $xy = 12$, to find the values of x and y.

OPERATION.

	$x + y = 7$	(1)
	$xy = 12$	(2)
Squaring (1),	$x^2 + 2xy + y^2 = 49$	(3)
Multiplying (2) by 4,	$4xy = 48$	(4)
Subtracting (4) from (3),	$x^2 - 2xy + y^2 = 1$	(5)
Evolving,	$x - y = \pm 1$	(6)
Equation (1),	$x + y = 7$	
Adding (6) and (1),	$2x = 8, \text{or } 6$	
Whence,	$x = 4, \text{or } 3$	
Subtracting (6) from (1),	$2y = 6, \text{or } 8$	
Whence,	$y = 3, \text{or } 4$	

What is said of the number of roots of homogeneous equations? What is said of the methods of solving equations under Case III.? Explain the operation.

Many complicated equations reduce to the sum and product of the two unknown quantities. After reaching that point, the work may conform to the above. It is evident, however, that such equations as the above belong under Case I. as well as Case III., and may therefore be solved by eliminating one of the unknown quantities from the original equations by substitution.

It must not be inferred that x and y are equal to each other; for when $x = 4$, $y = 3$, and when $x = 3$, $y = 4$. Whenever two simultaneous equations are symmetrical in their *signs*, as well as in other respects, it is evident that the letters may be exchanged without affecting the equation; hence the values of the letters must be interchangeable, and when the two values of one letter are found, the same values may be assigned to the other letter, the order being reversed.

2. Given $x^2 + y^2 = 25$, and $xy = 12$, to find the values of x and y.

OPERATION.

$x^2 + y^2 = 25$ (1)

$xy = 12$ (2)

Multiplying (2) by 2, $2xy = 24$ (3)

Adding (1) and (3), $x^2 + 2xy + y^2 = 49$ (4)

Subtracting (3) from (1), $x^2 - 2xy + y^2 = 1$ (5)

Extracting square root of (4), $x + y = \pm 7$ (6)

Extracting square root of (5), $x - y = \pm 1$ (7)

Adding (6) and (7), $2x = \pm 8, \text{ or } \pm 6$

Subtracting (7) from (6), $2y = \pm 6, \text{ or } \pm 8$

Whence, $x = \pm 4, \text{ or } \pm 3$

Also, $y = \pm 3, \text{ or } \pm 4$

It is evident that the above Example might be classed under Case II. as well as under Case III.; but the method here adopted gives a simpler solution.

3. Given $x^3 - y^3 = 19$, and $x^2 y - x y^2 = 6$, to find the values of x and y.

By what other method might the equations be solved? What is said of the relative values of x and y in such equations? Explain the second operation. By what other method might Example 2 be solved?

OPERATION.

$x^3 - y^3 = 19$ (1)

$x^2y - xy^2 = 6$ (2)

Multiplying (2) by 3, $3x^2y - 3xy^2 = 18$ (3)

Subtr. (3) from (1), $x^3 - 3x^2y + 3xy^2 - y^3 = 1$ (4)

Extr. cube root of (4), $x - y = 1$ (5)

Dividing (2) by (5), $xy = 6$ (6)

Squaring (5), $x^2 - 2xy + y^2 = 1$ (7)

Multiplying (6) by 4, $4xy = 24$ (8)

Adding (7) and (8), $x^2 + 2xy + y^2 = 25$ (9)

Extr. square root of (9), $x + y = \pm 5$ (10)

Equation (5), $x - y = 1$

Adding (5) and (10), $2x = 6, \text{or} -4$

Whence, $x = 3, \text{or} -2$

Subtracting (5) from (10), $2y = 4, \text{or} -6$

Whence, $y = 2, \text{or} -3$

As the original equations are not symmetrical in their *signs*, the values of x and y are not interchangeable.

4. Given $x + y = 4$, and $x^{-1} + y^{-1} = 1$, to find the values of x and y.
Ans. $\begin{cases} x = 2. \\ y = 2. \end{cases}$

NOTE. Remove negative exponents, apply Axiom 7, and solve like Example 1.

5. Given $x^3 + y^3 = 65$, and $x + y = 5$, to find the values of x and y.
Ans. $\begin{cases} x = 4, \text{ or } 1. \\ y = 1, \text{ or } 4. \end{cases}$

NOTE. Divide one equation by the other (Art. 87), and square the second.

6. Given $\frac{x^2}{y} + \frac{y^2}{x} = 9$, and $x + y = 6$, to find the values of x and y.
Ans. $\begin{cases} x = 4, \text{ or } 2. \\ y = 2, \text{ or } 4. \end{cases}$

Explain the third operation. Why are not the values of the two unknown quantities interchangeable?

7. Given $x^4+x^2y^2+y^4=931$, and $x^2+xy+y^2=49$, to find the values of x and y.

Ans. $\begin{cases} x=\pm 5, \text{ or } \pm 3. \\ y=\pm 3, \text{ or } \pm 5. \end{cases}$

Note. Divide one equation by the other.

8. Given $x+y=61$, and $x^{\frac{1}{2}}+y^{\frac{1}{2}}=11$, to find the values of x and y.

Ans. $\begin{cases} x=36, \text{ or } 25. \\ y=25, \text{ or } 36. \end{cases}$

Substitute v for $x^{\frac{1}{2}}$, and z for $y^{\frac{1}{2}}$, and the equations will become $v^2+z^2=61$ and $v+z=11$; from which the values of v and z, and consequently of x and y, are readily found. Or;

Subtract $x+y=61$ from the square of $x^{\frac{1}{2}}+y^{\frac{1}{2}}=11$, and square the result.

286. Sometimes one of the given equations, or some combination of the two given equations, takes the quadratic form, an expression containing both unknown quantities being the basis. (Arts. 276–278.)

1. Given $x^2+y^2+xy-2x-2y=9$, and $xy=6$, to find the values of x and y.

Ans. $\begin{cases} x=3, \text{ or } 2. \\ y=2, \text{ or } 3. \end{cases}$

After adding the second equation to the first, their sum may be put in the quadratic form, thus:

$$(x+y)^2-2(x+y)=15.$$

Completing the square, evolving, and reducing, we obtain

$$x+y=5, \text{ or } -3;$$

but as the latter value produces imaginary results, we use only the former.

2. Given $4xy=96-x^2y^2$, and $x+y=6$, to find the values of x and y.

Ans. $\begin{cases} x=4, \text{ or } 2, \text{ or } 3\pm\sqrt{21}. \\ y=2, \text{ or } 4, \text{ or } 3\mp\sqrt{21}. \end{cases}$

How may an equation containing two unknown quantities take the quadratic form?

NOTE. The signs ± and ∓, if used independently, would have the same signification; but when taken in connection, one is the reverse of the other. When x takes the upper sign, or $+$, y must also take the upper sign, or $-$; and when x takes the lower sign, or $-$, y must also take the lower sign, or $+$; that is, x and y must always take opposite signs. In the course of the operation, the sign ± is changed to ∓ whenever $+$ would be changed to $-$.

3. Given $\frac{x^2}{y^2}+\frac{4x}{y}=\frac{85}{9}$, and $x-y=2$, to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 5, \text{ or } \frac{17}{10}. \\ y = 3, \text{ or } -\frac{3}{10}. \end{cases}$$

287. Two equations, neither of which is strictly symmetrical in itself, may sometimes produce a symmetrical equation when properly combined.

Two equations which are not symmetrical in respect to the unknown quantities themselves, may be symmetrical in respect to some multiple or power of those unknown quantities; that is, the same multiple or power is the basis of the forms found in both equations.

Sometimes it is convenient to obtain one simple equation by means of the expedients used in Case III., and then complete the solution as in Case I.

1. Given $x^2+xy=60$, and $y^2+xy=84$, to find the values of x and y.

$$\text{Ans.} \begin{cases} x = \pm 5. \\ y = \pm 7. \end{cases}$$

Add the two equations, and the result is symmetrical. Extract the square root of the sum, and divide each equation by the result.

2. Given $x^2+9y^2=52$, and $x+3y=10$, to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 6, \text{ or } 4. \\ y = \frac{4}{3}, \text{ or } 2. \end{cases}$$

These equations are symmetrical in respect to x and $3y$. By substituting z for $3y$, each equation will become strictly symmetrical, and the values of x and z will be interchangeable.

How may equations not strictly symmetrical be brought under Case III.?

3. Given $x^{\frac{2}{3}} + y^{\frac{1}{2}} = 7$, and $x^{\frac{4}{3}} y = 144$, to find the values of x and y. Ans. $\begin{cases} x = \pm 8, \text{ or } \pm 3\sqrt{3}. \\ y = 9, \text{ or } 16. \end{cases}$

Substituting v for $x^{\frac{2}{3}}$, and z for $y^{\frac{1}{2}}$, the equations become $v + z = 7$, and $v^2 z^2 = 144$, from which the values of v and z are readily obtained. Replacing $x^{\frac{2}{3}}$ and $y^{\frac{1}{2}}$, the values of x and y are found.

After going through the operation as above, the student may take precisely the same steps, and find the values of $x^{\frac{2}{3}}$ and $y^{\frac{1}{2}}$, without the use of v and z.

NOTE. $x = \left(\frac{7 \pm \sqrt{97}}{2}\right)^{\frac{3}{2}}$ and $y = \left(\frac{7 \mp \sqrt{97}}{2}\right)^2$ may also be obtained from the equations last given.

4. Given $x^2 + 3xy = 54$, and $xy + 4y^2 = 115$, to find the values of x and y. Ans. $\begin{cases} x = \pm 3, \text{ or } \pm 36. \\ y = \pm 5, \text{ or } \mp \frac{23}{2}. \end{cases}$

Add the two equations together, and the result is a perfect square.

288. The following equations are to be solved by either of the methods already explained. As has already been shown, many of those which come under Case III. may also be classed under one of the first two Cases. Several solutions of the same set of equations are often possible, and the student should therefore seek to obtain the *best.*

EXAMPLES.

1. Given $\begin{Bmatrix} x^2 - y^2 = 24 \\ x + y = 6 \end{Bmatrix}$. Ans. $\begin{cases} x = 5. \\ y = 1. \end{cases}$

2. Given $\begin{Bmatrix} x + y = a \\ xy = b \end{Bmatrix}$. Ans. $\begin{cases} x = \dfrac{a \pm \sqrt{a^2 - 4b}}{2}. \\ y = \dfrac{a \mp \sqrt{a^2 - 4b}}{2}. \end{cases}$

3. Given $\left\{\begin{array}{l} x+y=a \\ x^2+y^2=c \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=\dfrac{a \pm \sqrt{2c-a^2}}{2}. \\ y=\dfrac{a \mp \sqrt{2c-a^2}}{2}. \end{array}\right.$

4. Given $\left\{\begin{array}{l} x+y=2 \\ x^{-1}+y^{-1}=2 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=1. \\ y=1. \end{array}\right.$

5. Given $\left\{\begin{array}{l} x^{-2}+y^{-2}=\frac{5}{9} \\ x^{-1}+y^{-1}=1 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=\frac{3}{2}, \text{ or } 3. \\ y=3, \text{ or } \frac{3}{2}. \end{array}\right.$

6. Given $\left\{\begin{array}{l} x^3-y^3=8 \\ x-y=2 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=2, \text{ or } 0. \\ y=0, \text{ or } -2. \end{array}\right.$

7. Given $\left\{\begin{array}{r} x^4+y^4=82 \\ xy=3 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=\pm 3, \text{ or } \pm 1. \\ y=\pm 1, \text{ or } \pm 3. \end{array}\right.$

Note. There are also the same number of imaginary roots.

8. Given $\left\{\begin{array}{l} x-y=8\left(\sqrt{x}-\sqrt{y}\right) \\ \sqrt{xy}=15 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x=25, \text{ or } 9. \\ y=9, \text{ or } 25. \end{array}\right.$

9. Given $\left\{\begin{array}{l} x+y=72 \\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=6 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=64, \text{ or } 8. \\ y=8, \text{ or } 64. \end{array}\right.$

10. Given $\left\{\begin{array}{r} x+y:x-y::13:5 \\ y^2+x=25 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x=9, \text{ or } -14\frac{1}{16}. \\ y=4, \text{ or } -6\frac{1}{4}. \end{array}\right.$

11. Given $\left\{\begin{array}{r} x+4y=14 \\ 4x-2y+y^2=11 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x=2, \text{ or } -46. \\ y=3, \text{ or } 15. \end{array}\right.$

12. Given $\left\{\begin{array}{r} x+3y=7 \\ x^2-3xy+3y^2=7 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x=1, \text{ or } 4. \\ y=2, \text{ or } 1. \end{array}\right.$

13. Given $\left\{\begin{array}{r} x^2 - 4y^2 = 9 \\ xy + 2y^2 = 18 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = \pm 5. \\ y = \pm 2. \end{array}\right.$

14. Given $\left\{\begin{array}{r} x^2 + xy + 2y^2 = 74 \\ 2x^2 + 2xy + y^2 = 73 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x = \pm 3, \text{ or } \mp 8. \\ y = \pm 5. \end{array}\right.$

15. Given $\left\{\begin{array}{r} x^2 + 2y^2 = 9 \\ xy + 2x^2 = 4 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x = \pm 1, \text{ or } \pm \frac{4}{3}\sqrt{2}. \\ y = \pm 2, \text{ or } \mp \frac{7}{6}\sqrt{2}. \end{array}\right.$

16. Given $\left\{\begin{array}{r} x^2 + y^2 + x + y = 18 \\ 2xy = 12 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x = 3, \text{ or } 2, \text{ or } -3 \pm \sqrt{3}. \\ y = 2, \text{ or } 3, \text{ or } -3 \mp \sqrt{3}. \end{array}\right.$

THEORY OF QUADRATIC EQUATIONS.

289. Every complete quadratic equation may be reduced to the form $x^2 + px = q$,

whose roots are $-\frac{p}{2} + \sqrt{q + \frac{p^2}{4}}$,

and $-\frac{p}{2} - \sqrt{q + \frac{p^2}{4}}$. (Art. 269, Ex. 5.)

It is evident that the sum of these roots is $-p$, and their product is $\frac{p^2}{4} - \left(q + \frac{p^2}{4}\right) = -q$.

Hence, when the coefficient of the first term is unity,

1. *The algebraic sum of the two roots of a quadratic equation is equal to the coefficient of the second term, with its sign changed.*

2. *The product of the two roots is equal to the second member, with its sign changed.*

What relation exists between the roots of a quadratic equation and the coefficient of the second term? Between the roots and the second member?

290. Using r and r' for the two roots of the quadratic equation $x^2 + px - q = 0,$

we have $x = r$, and $x = r'$,

or, $x - r = 0$, and $x - r' = 0$.

Multiplying the last two expressions together,

$$(x - r)(x - r') = 0,$$

or, $x^2 - (r + r')x + rr' = 0.$

But, by Art. 289, $r + r' = -p$, and $rr' = -q$;

hence, $x^2 + px - q = (x - r)(x - r') = 0.$

That is,

If all the terms of a quadratic equation be transposed to the first member, it may be resolved into the two binomial factors formed by subtracting each of the two roots of the equation from the unknown quantity.

EXAMPLES.

1. Resolve $x^2 - 4x + 3 = 0$ into binomial factors.

Ans. $(x - 3)(x - 1) = 0.$

A solution of the equation gives 3 and 1 as the roots.

2. Resolve $x^2 - \frac{x}{6} - \frac{1}{3} = 0$ into binomial factors.

Ans. $(x - \frac{2}{3})(x + \frac{1}{2}) = 0.$

3. Resolve $x^2 - 7x + 12 = 0$ into binomial factors.

4. Resolve $x^2 + 6x + 8 = 0$ into binomial factors.

291. The principle established in the last Article furnishes a method of resolving into factors any trinomial which contains the first and second powers of a letter or quantity. (Art. 90.) Such a trinomial is called a *quadratic expression.*

EXAMPLES.

1. Resolve $x^2 - 5x + 6$ into binomial factors.

Ans. $(x - 3)(x - 2.)$

How may a quadratic equation be factored? What is a quadratic expression?

Although this trinomial may have any value whatever, yet we find its factors by supposing it equal to 0, and obtaining the roots of the equation thus produced. The *factors* of the trinomial will remain the same, whatever the values of x, and of the trinomial.

2. Resolve $x^2 - \frac{x}{12} - \frac{5}{8}$ into binomial factors.

Ans. $(x - \frac{5}{6})(x + \frac{3}{4})$.

3. Resolve $x^2 + 3x - 28$ into binomial factors.

4. Resolve $z^2 + 18z + 80$ into binomial factors.

Note. The factors will be irrational or imaginary whenever the roots of the assumed equation are of that nature.

FORMATION OF EQUATIONS.

292. The principle established in Art. 290 also furnishes a method of forming a quadratic equation which shall have any two given roots.

RULE.

Subtract each of the given roots from the unknown quantity, and place the product of the two binomial factors equal to 0.

Examples.

1. What is the equation whose roots are 1 and -2?

OPERATION.

$$(x - 1)(x + 2) = x^2 + x - 2 = 0$$

Or, $$x^2 + x = 2$$

2. What is the equation whose roots are 4 and 5?

Ans. $x^2 - 9x = -20$.

3. What is the equation whose roots are 6 and 7?

How may a quadratic expression be factored? When will the factors be irrational or imaginary? Repeat the Rule for forming an equation having any two given roots. Explain the operation.

4. Form an equation whose roots shall be -1 and -2.
Ans. $x^2 + 3x = -2$.

5. Form an equation whose roots shall be 20 and -30.
Ans. $x^2 + 10x = 600$.

6. Form an equation whose roots shall be $1\frac{1}{2}$ and $-2\frac{4}{5}$.
Ans. $10x^2 + 13x = 42$.

7. Required the equation whose roots are a and b.
Ans. $x^2 - (a + b)x = -ab$.

8. Required the equation whose roots are $m + n$ and $m - n$.
Ans. $x^2 - 2mx = n^2 - m^2$.

NOTE. Most of the foregoing principles might be extended, with suitable modifications, to equations of a higher degree than the second. In general, the number of roots, and consequently of binomial factors, will correspond with the degree of the equation.

DISCUSSION OF THE GENERAL EQUATION.

293. If $q = 0$, in the general equation $x^2 + px = q$, the roots will become $-\frac{p}{2} \pm \frac{p}{2}$, or $-p$ and 0, and the equation may be considered a simple one. (Art. 269, Ex. 19, Note.)

If $p = 0$, the term px disappears, the roots become $+\sqrt{q}$ and $-\sqrt{q}$, and the equation is found to be a pure quadratic. Hence the principles stated in Arts. 289, 290, and 292 may be applied to both pure and affected quadratic equations.

294. The values of p and q in the general equation may be either positive or negative. If those letters be considered essentially positive, and the signs be expressed, we shall have *four forms.*

$$x^2 + px = q, \qquad x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}; \qquad (1)$$

If $q = 0$, what will be the effect on the general equation? What is said of pure quadratic equations? What are the four forms?

$$x^2 - px = q, \qquad x = \frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}; \qquad (2)$$

$$x^2 + px = -q, \qquad x = -\frac{p}{2} \pm \sqrt{-q + \frac{p^2}{4}}; \qquad (3)$$

$$x^2 - px = -q, \qquad x = \frac{p}{2} \pm \sqrt{-q + \frac{p^2}{4}}. \qquad (4)$$

295. As q is positive in the first and second forms, and negative in the third and fourth, the roots must have different signs in the first two forms, and the same sign in the last two. (Art. 289.) Moreover, since $\frac{p}{2}$ must be numerically greater than $\sqrt{-q + \frac{p^2}{4}}$, the signs of the roots in the last two forms must be controlled by the sign of $\frac{p}{2}$. Hence,

1. *In the first and second forms, one root is positive and the other negative.*

2. *In the third form, both roots are negative.*

3. *In the fourth form, both roots are positive.*

296. It is evident that the quantities under the radical sign of the roots in the first two forms can never be negative, and that they can be negative in the last two forms only when $\frac{p^2}{4}$ is numerically less than q. Hence,

1. *In the first and second forms, both roots are always real.*

2. *In the third and fourth forms, both roots are imaginary when the square of half the coefficient of x is numerically less than the second member; otherwise they are real.*

297. It is evident that the radical portion of the roots can never become 0 in the first two forms; but if $\frac{p^2}{4} = q$,

---

What will be the signs of the roots in each form? When will the roots be real, and when imaginary?

the radical portion will become 0 in the last two forms, and both roots will be $-\frac{p}{2}$, or $\frac{p}{2}$. Hence,

1. *In the first and second forms, the two roots are always numerically unequal.*

2. *In the third and fourth forms, the two roots are equal when the square of half the coefficient of x is numerically equal to the second member; otherwise they are unequal.*

NOTE. Examples to illustrate the foregoing principles, as well as a statement of some of the principles themselves, may be found in Art. 269.

PROBLEMS

LEADING TO AFFECTED QUADRATIC EQUATIONS.

298. The general principles involved in stating problems leading to quadratic equations are the same as those which have already been given in connection with simple equations.

The principles established in the Discussion of Problems (Arts. 179–184) are also equally applicable here; but we must note certain peculiarities, arising from the facts that every quadratic equation has *two roots,* and that those roots are sometimes *imaginary.*

1. The positive root of the equation is usually the true answer to the given problem. If there are two positive roots, there may be two answers to the problem, either of which conforms to the given conditions.

2. A negative result is sometimes the answer to another analogous problem, formed by attributing to the unknown quantity a quality directly opposite to that which has been attributed to it. As the algebraic mode

---

When will the roots be unequal, and when equal? What is said of the statement of problems leading to quadratic equations? What of the interpretation of results?

of expression is more general than ordinary language, the same equation often represents two analogous problems in this manner.

3. An imaginary result shows that the problem is an impossible one.

299. Some of the following problems require the use of but a single unknown quantity, others require the use of two, and others still may be solved by either method.

Some problems may also lead to either pure or affected quadratic equations, according to the notation assumed.

PROBLEMS.

1. A man buys a watch, which he sells again for $24, and finds that he loses as much per cent as the watch cost; required the price of the watch.

SOLUTION.

Let $x =$ the price in dollars.

Then $x =$ the loss per cent,

and $\frac{x}{100} \times x = \frac{x^2}{100} =$ his whole loss.

Therefore, $\frac{x^2}{100} = x - 24$

Or, $x^2 - 100x = -2400$

Completing square, $x^2 - 100x + 2500 = 100$

Whence, $x - 50 = \pm 10$

And, $x = 60$, or 40

The price was either $ 60 or $ 40, for each of these values satisfies all the conditions of the problem.

2. What number is that which exceeds the square of its fourth part by 3? Ans. 12, or 4.

What is said of the number of unknown quantities? Explain the solution of Problem 1.

3. Divide the number 10 into two parts whose product shall be 24.

SOLUTION.

Let $x =$ one part,

and $10 - x =$ the other part.

Then, $x(10 - x) = 24$

Or, $x^2 - 10x = -24$

Completing the square, $x^2 - 10x + 25 = 1$

Whence, $x - 5 = \pm 1$

And $x = 4$, or 6

Also, $10 - x = 6$, or 4

One part must be 4, and the other 6, and there is only one mode of dividing 10 so that the product of the two parts shall be 24; but it is immaterial *which* part is 4, and which is 6.

The same results may be obtained by the use of two unknown quantities, producing the symmetrical equations

$$x + y = 10, \text{ and } xy = 24.$$

4. A person bought a certain number of sheep for $80; if he had bought 4 more for the same sum, each sheep would have cost $1 less; required the number of sheep, and the price of each.

SOLUTION.

Let $x =$ the number of sheep.

Then $\frac{80}{x} =$ the price of each,

and $\frac{80}{x+4} =$ the price of each if he had bought 4 more.

Therefore, $\frac{80}{x+4} = \frac{80}{x} - 1$

Or, $80x = 80(x + 4) - x^2 - 4x$

Hence, $x^2 + 4x = 320$

Completing square, $x^2 + 4x + 4 = 324$

Explain the solution of Problem 3, Problem 4.

Whence, $x + 2 = \pm 18$

And $x = 16$, or -20

Also, $\frac{80}{x} = 5$, or -4

The negative results are not admissible, as answers to the above problem in its present form. (Art. 298.) The number of sheep was therefore 16, and the price of each $ 5.

If, in the above problem, "bought" be changed to *sold*, "4 more" to 4 *fewer*, and "$ 1 less" to $ 1 *more*, 20 and 4 will be the true answers. It will be well for the pupil to interpret the negative results in the problems which follow, whenever an obvious interpretation occurs.

5. Having sold a piece of goods for $ 56, I gained as much per cent as the whole cost me. How much did it cost? Ans. $ 40.

6. A person bought a lot of chickens for 96 cents, which he sold again at $13\frac{1}{2}$ cents a piece, and gained as much as one chicken cost him. What number did he buy? Ans. 8.

7. A printer, reckoning the cost of printing a book at so much per page, made the whole book come to $ 80. It turned out, however, that the book contained 5 pages more than he reckoned, and an abatement also was made of 50 cents per page. He received $ 67.50. How many pages did the book contain? Ans. 45 pages.

8. A company at a tavern had $ 8.75 to pay; but, before the bill was paid, two of them went away, when those who remained had, in consequence, 50 cents more to pay. How many persons were in the company at first? Ans. 7.

9. A sum of $ 1000 has to be divided equally among a number of persons; but two new claimants appearing, it is found that each person will receive $ 25 less than he expected. Required the original number of persons. Ans. 8.

Interpret the negative results.

10. The plate of a looking-glass is 18 inches by 12, and it is to be surrounded by a plain frame of uniform width, having a surface equal to that of the glass. Required the width of the frame. Ans. 3 inches.

11. Twenty persons contribute to send a donation of $48 to a benevolent society, one half of the whole being furnished in equal portions by the women, and the other half by the men; but each man gives a dollar more than each woman. How many are there of each sex, and what does each person contribute?

SOLUTION.

Let $x =$ number of women,
and $y =$ contribution of each in dollars.
Also, $20 - x =$ number of men,
and $y + 1 =$ contribution of each in dollars.
Then $xy =$ whole contrib. by the women,
and $(20 - x)(y + 1) =$ whole contrib. by the men.

Therefore, $xy = 24$ (1)
Also, $(20 - x)(y + 1) = 24$ (2)
From (1), $y = \frac{24}{x}$ (3)
From (2), $20y + 20 - xy - x = 24$ (4)
Subst. (3) in (4), $20\left(\frac{24}{x}\right) + 20 - 24 - x = 24$ (5)
Or, $\frac{480}{x} - x = 28$ (6)
Clearing of fractions, $x^2 + 28x = 480$ (7)
Completing the square, $x^2 + 28x + 196 = 676$ (8)
Evolving, $x + 14 = \pm 26$ (9)
Whence, $x = 12$, or -40
And $y = 2$, or $-\frac{3}{5}$
Also, $20 - x = 8$, or 60
And $y + 1 = 3$, or $\frac{2}{5}$

Explain the solution of Problem 11.

The negative values of x and y, although furnishing a solution of the equations, evidently do not belong to the problem, and no obvious interpretation occurs for them; consequently, the positive values of x and y are the only admissible results. Therefore, the number of women is 12, contributing 2 dollars each, and the number of men is 8, contributing 3 dollars each.

12. It is required to divide the number 40 into two such parts, that the sum of their squares shall be 818.
Ans. 23 and 17.

13. Divide the number 60 into two such parts, that their product shall be to the sum of their squares in the ratio of 2 to 5. Ans. 20 and 40.

The last two Problems, as well as some others, lead to pure quadratic equations, when we let $x - y$ and $x + y$ represent the numbers.

14. The fore wheel of a carriage makes 6 revolutions more than the hind wheel, in going 120 yards; but it is found that, if the circumference of each wheel be increased one yard, it will make only 4 revolutions more than the hind wheel, in the same distance; required the circumference of each wheel.

SOLUTION.

Let $x =$ circumference of hind wheel in yards.
and $y =$ circumference of fore wheel in yards.
Then $\frac{120}{x} =$ number of revolutions of hind wheel.
and $\frac{120}{y} =$ number of revolutions of fore wheel.

Therefore, by the Problem, $\frac{120}{x} = \frac{120}{y} - 6$ (1)

Or, $xy = 20x - 20y$ (2)

Also, by the Problem, $\frac{120}{x+1} = \frac{120}{y+1} - 4$ (3)

Explain the solution of Problem 14.

Therefore, $$30(y+1) = (x+1)(29-y) \tag{4}$$
Or, $$30y + 30 = 29x + 29 - xy - y \tag{5}$$
Transposing and uniting, $$xy = 29x - 31y - 1 \tag{6}$$
From (2) and (6), $$29x - 31y - 1 = 20x - 20y \tag{7}$$
Or, $$9x = 11y + 1 \tag{8}$$
Therefore, $$x = \frac{11y+1}{9} \tag{9}$$
Substituting (9) in (2), $$\frac{11y^2+y}{9} = \frac{220y+20}{9} - 20y \tag{10}$$
Reducing, $$11y^2 + y = 40y + 20 \tag{11}$$
Or, $$11y^2 - 39y = 20 \tag{12}$$
Whence, $$y = 4, \text{or} -\tfrac{5}{11}$$
And $$x = 5, \text{or} -\tfrac{4}{9}$$

The negative values of x and y being inadmissible, the circumference of the hind wheel is 5 yards, and that of the fore wheel is 4 yards.

If we take $\frac{5}{11}$ for the fore wheel, and $\frac{4}{9}$ for the hind wheel, the hind wheel must make 6 revolutions more than the fore wheel; and if each circumference be made equal to the difference between itself and unity, the fore wheel will make 4 revolutions more than the hind wheel.

15. A merchant buys two bales of cloth, each containing 80 yards, for $60. By selling the first at a gain of as much per cent as the second cost him per yard, in cents, and the second at a loss of as much per cent, he finds he has made a profit of $5 on the whole. Required the cost of each bale per yard.

Ans. First, 50 cents; second, 25 cents: or, first, 62½ cts.; second, 12½ cents.

16. There are two numbers whose sum multiplied by the greater gives 144, and whose difference multiplied by the less gives 14; what are the numbers?

Ans. 9 and 7.

17. A merchant bought as many bushels of corn as cost him $60, and, after reserving for his own use 15

bushels, sold the remainder for $ 54, and gained 10 cents a bushel; how many bushels did he buy?

Ans. 75 bushels.

18. The sum of the digits of a certain number is 15; and if 31 be added to their product, the sum will be equal to the number with its digits transposed. What is the number? Ans. 78.

19. In a purse containing 8 coins of gold and silver, each gold coin is worth as many dollars as there are silver coins, and each silver coin is worth as many cents as there are gold coins; and the whole is worth $ 15.15. How many are there of each?

Ans. 3 gold coins and 5 silver coins; or, 5 gold and 3 silver.

20. What are eggs a dozen, when two more for twelve cents lowers the price one cent per dozen?

Ans. 9 cents.

21. A farmer has inclosed a rectangular piece of land, containing 1 acre and 32 square rods, with 88 panels of fence, each 4 yards long; how many panels has he placed in each side of the rectangle?

Ans. 33 on one side, and 11 on the other.

22. There are four consecutive numbers, of which if the first two be taken for the digits of a number, that number is the product of the other two. What are the four numbers? Ans. 5, 6, 7, 8; or 1, 2, 3, 4.

23. A student traveled on a coach 6 miles into the country, and walked back at a rate 5 miles less per hour than that of the coach. He found that he was 50 minutes more in returning than in going. What was the speed of the coach? Ans. 9 miles per hour.

24. A gentleman sent a lad into the market to buy 12 cents' worth of peaches. The lad having eaten a couple, the gentleman paid at the rate of a cent for fif-

teen more than the market price. How many did the gentleman receive? Ans. 18.

25. A and B run a race. B, who runs slower than A by a mile in 5 hours, starts first by $2\frac{1}{2}$ minutes, and they get to the 5 mile stone together. Required their rates of running. Ans. A, 5 miles an hour; B, $4\frac{4}{5}$ miles.

26. A room is 40 feet long, and twice as broad as it is high. The cost of papering its walls, at $37\frac{1}{2}$ cents per yard, is $71.50. Required the height of the room, no allowance being made for doors or windows.

Ans. 13 feet.

27. A mirror is in the shape of a double square. The cost of the glass, at $1.25 a square foot, exceeds the cost of the frame, at 75 cents a linear foot, measured on the inside of the frame, by $22. Required the dimensions of the glass. Ans. 8 feet by 4 feet.

28. Two detachments of soldiers being ordered to a station at the distance of 39 miles from their present quarters, begin their march at the same time; but one party, by traveling $\frac{1}{4}$ of a mile an hour faster than the other, arrives there an hour sooner. Required their rates of marching. Ans. $3\frac{1}{4}$ and 3 miles per hour.

29. Find two numbers whose sum is 6 and whose product is 10. Ans. Impossible.

The imaginary expressions $3+\sqrt{-1}$ and $3-\sqrt{-1}$ alone answer the conditions, and these are readily obtained.

30. There are two lots, each of which is an exact square; it requires 200 rods of fence to inclose both, and their contents are 1300 square rods. What is the value of each at $2.25 per square rod?

Ans. The smaller, $900; the larger, $2025.

31. A grocer sold 80 pounds of tea, and 100 pounds of coffee, for $65; but he sold 60 pounds more of coffee

for $20 than he did of tea for $10. What was the price of 1 pound of each?

Ans. Tea, 50 cents; coffee, 25 cents.

32. Two farmers drove to market 100 sheep between them, and returned with equal sums. If each of them had sold his sheep at the same price that the other actually did, the one would have returned with $180, and the other with $80. At what price per sheep did they sell, respectively, and how many sheep had each?

Ans. At $2 and $3 per sheep; the one had 60 sheep, and the other 40.

RATIO AND PROPORTION.

300. The Ratio of one quantity to another of the same kind is the quotient arising from dividing the first quantity by the second. (Art. 162.)

Thus, the ratio of a to b is $\frac{a}{b}$, or $a : b$.

301. The Terms of a ratio are the two quantities required to form it.

The first term is called the *antecedent* of the ratio, and the second, the *consequent*.

Thus, in the ratio of a to b, or $a : b$, a and b are the terms, of which a is the antecedent and b the consequent.

302. A Direct Ratio is one in which the antecedent is divided by the consequent.

An Inverse Ratio is one in which the consequent is divided by the antecedent.

Thus, the direct ratio of 6 to 3, or $6 : 3$, is $\frac{6}{3}$, or 2, and the inverse ratio is $\frac{3}{6}$, or $\frac{1}{2}$.

Define Ratio. The Terms of a ratio. A Direct Ratio. An Inverse Ratio.

NOTE. When the kind of ratio is not mentioned, the interpretation is understood to be that of the direct ratio. This method has the almost universal sanction of mathematicians in all countries. The so-called French interpretation is not that of the modern French mathematicians.

303. A PROPORTION is an equality of ratios.

Four quantities are in proportion when the ratio of the *first* to the *second* is the same as that of the *third* to the *fourth*.

Thus, the ratios $a : b$ and $c : d$, if equal to each other, form a proportion, when written

$$a : b = c : d, \text{ or } a : b : : c : d.$$

304. The TERMS of a proportion are the terms of the ratios forming the proportion.

305. The ANTECEDENTS in a proportion are the first terms of its ratios, or the first and third terms of the proportion.

The CONSEQUENTS in a proportion are the last terms of its ratios, or the second and fourth terms of the proportion.

Thus, a and c are the antecedents, and b and d the consequents, in the proportion

$$a : b : : c : d.$$

306. The EXTREMES of a proportion are its first and last terms.

The MEANS of a proportion are its second and third terms.

Thus, a and d are the extremes, and b and c the means, in the proportion

$$a : b : : c : d.$$

307. A COUPLET consists of the two terms of a ratio.

Define a Proportion. When are four quantities in proportion? Define the Terms of a proportion. The Antecedents. The Consequents. The Extremes. The Means. A Couplet.

308. A PROPORTIONAL is any one of the terms of a proportion.

Thus, the fourth term, d, is the fourth proportional to a, b, and c, taken in their order in the proportion

$$a : b :: c : d.$$

309. A MEAN PROPORTIONAL between two quantities is either of the two means, when they are the same quantity. Thus, in the proportion

$$a : b :: b : c,$$

b is a mean proportional between a and c.

310. A CONTINUED PROPORTION is one in which each consequent is the same as the next antecedent.

Thus, in the proportion

$$a : b :: b : c :: c : d :: d : e,$$

the quantities a, b, c, d, and e are said to be in continued proportion.

311. Quantities are in proportion by ALTERNATION, when antecedent is compared with antecedent, and consequent with consequent.

312. Quantities are in proportion by INVERSION, when each antecedent is made a consequent, and each consequent an antecedent.

313. Quantities are in proportion by COMPOSITION, when the sum of antecedent and consequent is compared with either antecedent or consequent.

314. Quantities are in proportion by DIVISION, when the difference of antecedent and consequent is compared with either antecedent or consequent.

Define a Proportional. A Mean Proportional. A Continued Proportion. When are quantities in proportion by Alternation? When by Inversion? When by Composition? When by Division?

THEOREMS RELATING TO PROPORTION.

THEOREM I.

315. *In every proportion, the product of the extremes is equal to the product of the means.*

Let $a : b :: c : d,$

or, $\frac{a}{b} = \frac{c}{d}.$

Clearing of fractions, $ad = bc.$

Hence, *if three terms of a proportion be given, the fourth may be found.*

Let $a : b :: c : x.$

Then, $ax = bc;$

whence, $x = \frac{bc}{a}.$

THEOREM II.

316. *If the product of two quantities be equal to the product of two others, two of them may be made the extremes and the other two the means of a proportion.*

Let $ad = bc.$

Dividing by bd and reducing, $\frac{a}{b} = \frac{c}{d},$

or, $a : b :: c : d.$

THEOREM III.

317. *If three quantities be in continued proportion, the product of the two extremes is equal to the square of the mean.*

Let $a : b :: b : c.$

Then, by Theo. I., $ac = bb = b^2.$

Demonstrate Theorem I. Show that the fourth term of a proportion may be found when three are given. Demonstrate Theorem II. Theorem III.

THEOREM IV.

318. *If four quantities be in proportion, they will be in proportion by* ALTERNATION.

Let $a : b :: c : d$,

or, $\frac{a}{b} = \frac{c}{d}$.

Multiplying by $\frac{b}{c}$, and reducing, $\frac{a}{c} = \frac{b}{d}$,

or, $a : c :: b : d$.

THEOREM V.

319. *If four quantities be in proportion, they will be in proportion by* INVERSION.

Let $a : b :: c : d$.

Then, by Theo. I., $ad = bc$, or $bc = ad$;

whence, by Theo. II., $b : a :: d : c$.

THEOREM VI.

320. *If four quantities be in proportion, they will be in proportion by* COMPOSITION.

Let $a : b :: c : d$;

then, $\underline{a + b : b :: c + d : d}$.

For, by equality of ratios, $\frac{a}{b} = \frac{c}{d}$.

Adding 1 to each side, $\frac{a}{b} + 1 = \frac{c}{d} + 1$,

or, $\frac{a+b}{b} = \frac{c+d}{d}$;

whence, $a + b : b :: c + d : d$.

THEOREM VII.

321. *If four quantities be in proportion, they will be in proportion by* DIVISION.

Demonstrate Theorem IV. Theorem V. Theorem VI. Theorem VII.

Let $a : b :: c : d;$

then, $\underline{a - b : b :: c - d : d.}$

For, by equality of ratios, $\frac{a}{b} = \frac{c}{d}.$

Subtracting 1 from each side, $\frac{a}{b} - 1 = \frac{c}{d} - 1,$

or, $\frac{a-b}{b} = \frac{c-d}{d};$

whence, $a - b : b :: c - d : d.$

THEOREM VIII.

322. *If four quantities be in proportion, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

Let $a : b :: c : d;$

Then, $\underline{a + b : a - b :: c + d : c - d}$

By Theo. VI., $\frac{a+b}{b} = \frac{c+d}{d},$

and by Theo. VII., $\frac{a-b}{b} = \frac{c-d}{d};$

therefore, $\frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d},$

or, $\frac{a+b}{a-b} = \frac{c+d}{c-d},$

whence, $a + b : a - b :: c + d : c - d.$

THEOREM IX.

323. *Quantities which are proportional to the same quantities are proportional to each other.*

Let $a : b :: e : f,$

and $c : d :: e : f;$

then $\underline{a : b :: c : d.}$

For, by equality of ratios, $\frac{a}{b} = \frac{e}{f},$

Demonstrate Theorem VIII. Theorem IX.

and $\frac{c}{d} = \frac{e}{f}$.

Therefore, by Art. 38, Ax. 7, $\frac{a}{b} = \frac{c}{d}$,

or, $a : b :: c : d$.

THEOREM X.

324. *If any number of quantities are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $a : b :: c : d :: e : f$;

then $\underline{a : b :: a + c + e : b + d + f}$.

For, by Theo. I., $ad = bc$,

and $af = be$;

also, $\underline{ab = ba}$.

Adding, $ab + ad + af = ba + bc + be$,

or, $a(b + d + f) = b(a + c + e)$;

whence, by Theo. II., $a : b :: a + c + e : b + d + f$.

THEOREM XI.

325. *When four quantities are in proportion, if the first and second be multiplied or divided by the same quantity, as also the third and fourth, the resulting quantities will be proportional.*

Let $a : b :: c : d$;

then, $\underline{ma : mb :: nc : nd}$.

For, by equality of ratios, $\frac{a}{b} = \frac{c}{d}$,

and $\frac{ma}{mb} = \frac{nc}{nd}$,

or, $ma : mb :: nc : nd$.

In like manner, $\frac{a}{m} : \frac{b}{m} :: \frac{c}{n} : \frac{d}{n}$.

Either m or n may be made equal to unity; that is, either couplet may be multiplied or divided without multiplying or dividing the other.

Demonstrate Theorem X. Theorem XI.

THEOREM XII.

326. *When four quantities are in proportion, if the first and third be multiplied or divided by the same quantity, as also the second and fourth, the resulting quantities will be proportional.*

Let $a : b :: c : d$;

then, $ma : nb :: mc : nd$.

For, by equality of ratios, $\frac{a}{b} = \frac{c}{d}$;

therefore, $\frac{ma}{b} = \frac{mc}{d}$,

and, $\frac{ma}{nb} = \frac{mc}{nd}$,

whence, $ma : nb :: mc : nd$.

In like manner, $\frac{a}{m} : \frac{b}{n} :: \frac{c}{m} : \frac{d}{n}$.

Either m or n may be made equal to unity.

THEOREM XIII.

327. *If there be two sets of proportional quantities, the products of the corresponding terms will be proportional.*

Let $a : b :: c : d$,

and $e : f :: g : h$;

then, $ae : bf :: cg : dh$.

For, by equality of ratios, $\frac{a}{b} = \frac{c}{d}$,

and $\frac{e}{f} = \frac{g}{h}$.

By multiplication, $\frac{ae}{bf} = \frac{cg}{dh}$,

or, $ae : bf :: cg : dh$.

THEOREM XIV.

328. *If four quantities be in proportion, like powers or roots of these quantities will be proportional.*

Demonstrate Theorem XII. Theorem XIII. Theorem XIV.

Let $a : b :: c : d;$

then $a^n : b^n :: c^n : d^n,$

and $a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}.$

For, by equality of ratios, $\frac{a}{b} = \frac{c}{d};$

raising to nth power, $\frac{a^n}{b^n} = \frac{c^n}{d^n},$

extracting the nth root, $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}};$

whence, $a^n : b^n :: c^n : d^n,$

and $a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}.$

PROBLEMS IN PROPORTION.

329. By means of the foregoing theorems, proportions may often be much simplified before changing them to equations.

1. Find a number to which if 3, 8, and 17 be severally added, the second sum shall be to the first as the third is to the second.

SOLUTION.

Let $x =$ the number.

Then $x + 8 : x + 3 :: x + 17 : x + 8$ (1)

From (1), by Theo. VII., $5 : x + 3 :: 9 : x + 8$ (2)

By Theo. I., $9x + 27 = 5x + 40$ (3)

Reducing, $4x = 13$ (4)

Whence, $x = 3\frac{1}{4}$, number required.

2. The sum of two numbers is 35, and their product is to the sum of their squares as 12 to 25; what are the numbers?

Explain the solution of Problem 1.

SOLUTION.

Let x and y represent the numbers.

Then, $$x + y = 35 \qquad (1)$$
and $$xy : x^2 + y^2 :: 12 : 25 \qquad (2)$$
By Theo. XII., $$2xy : x^2 + y^2 :: 24 : 25$$
By Theo. VIII., $$x^2 + 2xy + y^2 : x^2 - 2xy + y^2 :: 49 : 1$$
By Theo. XIV., $$x + y : x - y :: 7 : 1$$
By Theo. VIII., $$2x : 2y :: 8 : 6$$
By Theo. XI., $$x : y :: 4 : 3$$
By Theo. I., $$y = \frac{3x}{4}$$
Substituting in (1), and reducing, $$x = 20$$
Also, $$y = 15$$

3. The last three terms of a proportion being 4, 6, and 8, what is the first term?

4. If 3, x, and 1083, are in continued proportion, what is the value of x? Ans. $x = 57$.

5. If $a + x : a - x :: 11 : 7$, what is the ratio of a to x? Ans. $9 : 2$.

6. Triangles are to each other as the products of their bases by their altitudes. The bases of two triangles are to each other as 17 to 18, and their altitudes as 21 to 23; required the ratio of the triangles. Ans. 119 : 138.

7. A quantity of milk is increased by water in the ratio of 5 : 4, and then 3 gallons are sold; the rest, being mixed with 3 quarts of water, is increased in the ratio of 7 : 6. How many gallons of milk were there at first? Ans. 6 gallons.

8. A and B speculate with different sums. A gains \$100, B loses \$50, and now A's stock is to B's as 4 to 3; but had A lost \$50, and B gained \$100, then A's stock would have been to B's as 5 to 9. Required the stock of each. Ans. A's, \$300; B's, \$350.

Explain the solution of Problem 2.

9. The product of two numbers is 15, and the sum of their squares is to the difference of their squares as 17 to 8. What are the numbers? Ans. 5 and 3.

10. A and B have made a bet, each staking a sum of money proportional to all the money he has. If A wins, he will have double what B will have; but if he loses, B will have three times what A will have. All the money between them being $ 168, determine the circumstances.

Ans. A has $ 72, and B has $ 96; each stakes $\frac{5}{12}$ of his money.

SERIES.

330. A Series is a succession of terms, so related that each may be derived from one or more preceding ones, in accordance with some fixed law.

The *Terms* of a series are the quantities of which it is formed.

The *Extremes* of a series are the first and last terms.

The *Means* of a series are the terms between the extremes.

ARITHMETICAL PROGRESSION.

331. An Arithmetical Progression is a series that increases or decreases from term to term by a *common difference.*

332. The progression may be considered as formed by the continual *addition* of the common difference; therefore, when the series is *increasing,* the common difference will be *positive,* and when *decreasing,* it will be *negative.* Thus,

---

Define a Series. The Terms of a series. The Extremes. The Means. Arithmetical Progression. How may the progression be considered as formed?

1, 3, 5, 7, 9, 11, 13, etc.

is an *increasing* arithmetical progression, in which the common difference is $+2$; and

19, 17, 15, 13, 11, 9, 7, etc.

is a *decreasing* arithmetical progression, in which the common difference is -2.

333. In arithmetical progression, if we regard the number of terms as limited, there will be five elements for consideration:—

1. The first term.
2. The last term.
3. The number of terms.
4. The common difference.
5. The sum of the terms.

These are so related to each other, that, any *three* of them being given, the other *two* may be readily determined.

CASE I.

334. Given the first term, common difference, and number of terms, to find the last term.

Let a denote the first term, d the common difference, n the number of terms, and l the last term; then the progression will be

$$a,\ (a+d),\ (a+2d),\ (a+3d),\ (a+4d),\ \&c.$$

That is, the coefficient of d in any term is one less than the number of that term in the series; consequently, the nth or last term will be

$$a+(n-1)d.$$

Whence, putting l for the nth term, we have

$$l=a+(n-1)d,$$

in which d is either positive or negative, according as the series is an increasing or a decreasing one.

How many elements are there for consideration? How many must be given? Demonstrate the formula for finding the last term.

Hence the following

RULE.

To the first term add the product of the common difference by the number of terms less one.

EXAMPLES.

1. If the first term is 5, the common difference 3, and the number of terms 20, what is the last term?

Ans. 62.

2. If the first term is 4, the common difference 5, and the number of terms 30, what is the last term?

Ans. 149.

3. When the first term is 10 and the common difference -2, what is the fifth term? Ans. 2.

4. When the first term is -6 and the common difference $\frac{1}{2}$, what is the fifteenth term? Ans. 1.

5. If the first term is 15, the common difference -3, and the number of terms 6, what is the last term?

Ans. 0.

CASE II.

335. Given the first term, common difference, and number of terms, to find the sum of the terms.

Let a denote the first term, d the common difference, n the number of terms, l the last term, and S the sum of the terms. Then,

$$S = a + (a + d) + (a + 2d) + \dots\dots + l,$$

or, writing the terms in the reverse order,

$$S = l + (l - d) + (l - 2d) + \dots\dots + a.$$

Therefore, by adding these equations, term by term,

$$2S = (a + l) + (a + l) + (a + l) + \dots\dots + (a + l).$$

Repeat the Rule. Demonstrate the formula for finding the sum of an arithmetical series.

Here $(a + l)$ is taken as many times as there are terms, or n times; whence,

$$2S = n(a + l), \text{ or } S = \tfrac{1}{2}n(a + l).$$

Hence the following

RULE.

Multiply the sum of the extremes by half the number of terms.

NOTE. It will be readily perceived from the foregoing, that the sum of any two terms equidistant from the extremes is equal to the sum of the extremes.

EXAMPLES.

1. Find the sum of an arithmetical series, of which the first term is 3, the common difference 2, and the number of terms 20. Ans. 440.

2. If the first term is 7, the common difference -4, and the number of terms 6, what is the sum of the terms? Ans. -18.

3. Required the sum of the series $\frac{1}{2} + 1\frac{1}{2} + 2\frac{1}{2} +$ &c., to twenty terms. Ans. 200.

4. If the first term is 5, the last term 62, and the number of terms 20, what is the sum of the terms? Ans. 670.

5. The first term of an arithmetical series is $-3\frac{1}{3}$, the common difference $\frac{1}{5}$; required the sum of twenty-one terms. Ans. -28.

CASE III.

336. Given any three of the five elements of an arithmetical progression, to find either of the others.

Repeat the Rule. The Note.

The formulas established in Arts. 334, 335,

$$l = a + (n - 1)\,d, \qquad (1)$$
$$S = \tfrac{1}{2}n\,(a + l), \qquad (2)$$

are *fundamental* ones; and, since they constitute two independent equations, together containing all the five elements of an arithmetical progression, when any three of these are given, the other two may be readily determined. Thus, from these two equations, we deduce,—

1. The formulas for the first term:

$$a = l - (n - 1)\,d. \qquad (3)$$
$$a = \frac{2S}{n} - l. \qquad (4)$$

2. The formulas for the common difference:

$$d = \frac{l - a}{n - 1}. \qquad (5)$$
$$d = \frac{2S - 2an}{n\,(n - 1)}. \qquad (6)$$

3. The formulas for the number of terms:

$$n = \frac{l - a}{d} + 1. \qquad (7)$$
$$n = \frac{2S}{l + a}. \qquad (8)$$

In like manner it may be shown that twenty cases may arise, admitting of solution by some transposition or combination of formulas (1) and (2).

Note. Each formula must contain four elements. If neither of the fundamental formulas contains the required four, the superfluous one may be eliminated by combining the two formulas.

Hence the following

RULE.

Substitute in the fundamental formulas, or such as may be deduced from them, the given quantities, and reduce the result.

Repeat the fundamental formulas. Give the formulas for the first term. For the common difference. For the number of terms. How many cases may arise? Repeat the Rule.

EXAMPLES.

1. Required the first term, when the last term is 62, the common difference 3, and the number of terms 20.

Ans. 5.

2. Required the common difference, when the last term is 149, the first term 4, and the number of terms 30.

Ans. 5.

3. Required the number of terms, when the last term is 1, the first term -6, and the common difference $\frac{1}{2}$.

Ans. 15.

4. Required the first term, when the sum of the terms is 99, the number of terms 9, and the last term 19.

Ans. 3.

5. When the last term is 2, the first term 10, and the number of terms 5, what is the common difference?

Ans. -2.

CASE IV.

337. Given two terms, to insert any number of arithmetical means between them.

The terms between any other two terms of an arithmetical progression are called *arithmetical means.*

One mean between two terms is half their sum.

For, the number of terms is 3; therefore, the mean is $a + d$, and the last term is

$$l = a + 2d.$$

Hence, $$l + a = 2a + 2d,$$

or, $$a + d = \frac{l + a}{2}.$$

That is, $\frac{a + l}{2}$ is the arithmetical mean between a and l.

Let it now be required to insert any number, m, of arithmetical means between two given terms, a and l, the common difference still being denoted by d.

Define an arithmetical mean. Demonstrate the method of finding it.

The common difference added to the given first term will evidently give the first arithmetical mean, the common difference added to the first mean will give the second, and the m required means will be

$$a + d, \quad a + 2d, \quad a + 3d, \ldots\ldots a + md.$$

Hence the following

RULE.

Add the common difference to the given first term for the first arithmetical mean, add it to the first mean for the second mean, and so on.

EXAMPLES.

1. Find the arithmetical mean between 6 and 20.

Ans. 13.

2. Insert two arithmetical means between 5 and 14.

Ans. 8 and 11.

NOTE. The number of *terms* is evidently 4; hence, $d = \frac{14 - 5}{3}$.

3. Find the arithmetical mean between $\frac{1}{4}$ and $\frac{1}{9}$.

Ans. $\frac{13}{72}$.

4. Insert three arithmetical means between 1 and 3.

Ans. $1\frac{1}{2}$, 2, $2\frac{1}{2}$.

5. Find the arithmetical mean between $\frac{a}{2}$ and $\frac{b}{2}$.

Ans. $\frac{1}{4}(a + b)$.

6. Insert two arithmetical means between $\frac{1}{3}$ and $\frac{11}{6}$.

Ans. $\frac{5}{6}$ and $\frac{4}{3}$.

7. Insert two arithmetical means between x and y.

Ans. $\frac{2x + y}{3}$ and $\frac{x + 2y}{3}$.

Demonstrate the method of inserting any number of means between two given terms.

PROBLEMS.

338. Some of the following problems can be solved at once by means of the preceding rules, while others require an application of the principles of arithmetical progression in the course of an ordinary algebraic solution.

1. When a clock strikes the hours only, how many strokes does it make in 12 hours? Ans. 78.

2. Required the 15th term in the series $\frac{1}{3}$, $\frac{2}{3}$, 1, etc. Ans. 5.

3. Required the sum of 20 terms of the series, 15, 11, 7, etc. Ans. —460.

4. A certain debt was discharged in 25 weeks, by paying \$2 the first week, \$5 the second, \$8 the third, and so on. What was the amount of the debt? Ans. \$950.

5. Insert five arithmetical means between $\frac{1}{2}$ and $-\frac{1}{2}$. Ans. $\frac{1}{3}$, $\frac{1}{6}$, 0, $-\frac{1}{6}$, $-\frac{1}{3}$.

6. What is the common difference when the first term is 1, the last 50, and the sum of the terms 204? Ans. 7.

Eliminate n from the fundamental formulas, or find its value by (8), and substitute in (5).

7. Find how many terms there are in the series whose first term is 3, last term 7, and sum of the terms 25. Ans. 5.

8. A person saves \$20 a year, which he places at interest at 4 per cent., simple interest. To how much do his savings amount, with interest, in 20 years? Ans. \$552.

9. The sum of 8 terms of an arithmetical progression is 140, and the 8th term is 7. Required the series. Ans. 28, 25, 22, etc.

10. The annual expenses of a person, whose only source

of income is an entailed estate yielding \$3300 a year, are \$5300. He is therefore obliged to borrow \$2000 a year, at the rate of 10 per cent, the high rate of interest being requisite to cover insurance on his life. In how many years will he be ruined, reckoning the simple interest which accumulates? Ans. 11 years.

At the end of x years, his yearly debts will constitute an arithmetical progression, whose extremes are

$$2000 \text{ and } 2000\left(1+\frac{x-1}{10}\right).$$

The whole amount of his debts, or the sum of the terms, will then be

$$\frac{2000x}{2}\left(2+\frac{x-1}{10}\right)=100x^2+1900x.$$

By the conditions of the problem, the interest on this must be equal to the income from his estate, or

$$\frac{100x^2+1900x}{10}=3300.$$

11. A hare runs at the rate of 8 feet per second. A greyhound pursues her, starting from a distance of 60 feet, and running 8 feet the first second, $8\frac{1}{2}$ the second, 9 the third, etc. In how many seconds will he catch the hare? Ans. 16.

12. There is a series of terms in arithmetical progression, of which the sum of the first two terms is $2\frac{1}{2}$, and the 4th term is $2\frac{1}{2}$. What is the series?

Ans. 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$.

Let x = the first term, and y = the common difference.

13. A heavy body, falling from rest and unobstructed, passes through a space of $16\frac{1}{12}$ feet, nearly, in the first second of time, and afterwards in each succeeding second $32\frac{1}{6}$ feet more than in the second immediately preceding. Now a heavy body fell from the car of a balloon, and it was ascertained to have been exactly 20 seconds before it struck the earth. What was the height of the balloon, supposing the resistance of the air not worth reckoning?

Ans. 1.22 miles.

GEOMETRICAL PROGRESSION.

339. A GEOMETRICAL PROGRESSION is a series, each term of which is equal to the preceding one, multiplied by a constant factor.

The constant factor is called the *ratio* of the progression.

340. The successive terms of the progression may be considered as derived from the first by continually *multiplying* it by the ratio; therefore, if the first term is positive, the series is *increasing* when the ratio is *greater* than 1, but the series is *decreasing* when the ratio is *less* than 1. Thus,

$$3, \quad 6, \quad 12, \quad 24, \quad 48, \quad 96, \quad 192, \text{ etc.}$$

is an *increasing* geometrical progression, in which the first term is 3, and the ratio 2; and

$$27, \quad 9, \quad 3, \quad 1, \quad \tfrac{1}{3}, \quad \tfrac{1}{9}, \quad \tfrac{1}{27}, \text{ etc.}$$

is a *decreasing* geometrical progression, in which the ratio is $\frac{1}{3}$.

341. The terms between any other two terms of a geometrical progression are called *geometrical means.*

342. In geometrical progression, if we regard the number of terms as limited, there will be five elements for consideration:—

1. The first term.
2. The last term.
3. The number of terms.
4. The ratio.
5. The sum of the terms.

Define Geometrical Progression. The ratio of the progression. How may the successive terms be considered? When is the progression increasing, and when decreasing? What are geometrical means? How many elements in a geometrical progression?

These are so related to each other, that, any *three* of them being given, the other *two* may be readily determined.

CASE I.

343. Given the first term, the ratio, and the number of terms, to find the last term.

Let a denote the first term, r the ratio, and n the number of terms; then the successive terms of the series will be

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \ldots\ldots ar^{n-1}.$$

That is, the given first term is a factor of each of the terms, and the exponent of r in the *second* term is 1, in the *third* term 2, in the *fourth* term 3, and so on, to the nth term, in which it is $n-1$.

Therefore, if l denote the last term, we shall have

$$l = ar^{n-1}.$$

Hence the following

RULE.

Multiply the first term by the ratio raised to a power whose exponent is one less than the number of terms.

EXAMPLES.

1. Find the last term of a series whose first term is 5, ratio 4, and number of terms 8. Ans. 81920.

2. Find the 7th term of a series whose first term is 28672, and ratio $\frac{1}{4}$. Ans. 7.

3. The first term of a geometrical progression is 5, the ratio 4, and the number of terms 9. What is the last term? Ans. 327680.

4. Required the 6th term of the series whose first term is 100, and ratio $\frac{2}{3}$. Ans. $13\frac{41}{243}$.

5. Required the 9th term of the series 3, 6, 12, etc. Ans. 768.

Demonstrate the formula for finding the last term. Repeat the Rule.

CASE II.

344. Given the first term, the ratio, and the number of terms, to find the sum of the terms.

Let a denote the first term, r the ratio, n the number of terms, and S the sum of the terms.

Then, $$S = a + ar + ar^2 + ar^3 \ldots\ldots + ar^{n-2} + ar^{n-1} \quad (1)$$

Multiplying by r,
$$rS = ar + ar^2 + ar^3 + ar^4 \ldots\ldots + ar^{n-1} + ar^n \quad (2)$$

Subtracting (1) from (2), and factoring,
$$(r-1)S = a(r^n - 1). \quad (3)$$

Therefore, $$S = \frac{a(r^n-1)}{r-1}. \quad (4)$$

Again, if l denote the last term, by Case I.,
$$l = ar^{n-1}. \quad (5)$$

Multiplying by r,
$$rl = ar^n. \quad (6)$$

Substituting the value of ar^n in (4),
$$S = \frac{rl-a}{r-1}. \quad (7)$$

Hence the

RULE.

Multiply the last term by the ratio, subtract the first term, and divide the remainder by the ratio less 1.

NOTE. If the last term is not given, it may be found by Case I.; or, formula (4) may be used instead of formula (7).

EXAMPLES.

1. Find the sum of a geometrical series whose first term is 1, ratio 2, and last term 1024. Ans. 2047.

2. Find the sum of a series whose first term is 6, ratio 4, and number of terms 8. Ans. 131070.

Demonstrate the formulas for finding the sum of the terms. Repeat the Rule. The Note.

3. If the first term of a series is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the last term $\frac{1}{162}$, what is the sum of the terms?

Ans. $\frac{121}{162}$.

4. Required the sum of the series 1, 3, 9, 27, etc., continued to 12 terms. Ans. 265720.

5. Required the sum of the series 4, 2, 1, etc., to 16 terms. Ans. $8\left(1-\frac{1}{2^{16}}\right)=8-\frac{1}{8192}$.

345. The limit to which the sum of a decreasing geometrical series approaches, as the number of terms becomes larger and larger, is called the *sum of the series to infinity.*

When r is less than 1, to prevent the terms of the fraction in equation (4), Art. 344, from becoming negative, that formula may be placed under the equivalent form (Art. 121),

$$S=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

Now when the number of terms, n, becomes infinitely great, or equal to ∞, the fraction $\frac{ar^n}{1-r}$ must become infinitely small, or equal to 0, and may be rejected. Hence, when the number of terms in a decreasing geometrical series is infinite,

$$S=\frac{a}{1-r}.$$

That is, *the sum of the terms of a decreasing geometrical series to infinity is equal to the first term divided by* 1 *less the ratio.*

1. Find the sum of 4, 2, 1, $\frac{1}{2}$, etc., to infinity.

Ans. 8.

2. Find the sum of the infinite series $\frac{2}{7}$, $\frac{6}{35}$, $\frac{18}{175}$, etc.

Ans. $\frac{5}{7}$.

3. What is the sum of the series $.\dot{7}\dot{9}$, to infinity?

Ans. $\frac{79}{99}$.

What is meant by the sum of a series to infinity? Demonstrate the formula for obtaining it. To what is the sum equal?

4. What is the sum of 1, $\frac{1}{10}$, $\frac{1}{100}$, etc., to infinity?

Ans. $1\frac{1}{9}$.

5. Find the sum of 1, $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$, etc., to infinity.

Ans. $\frac{x}{x-1}$.

6. Required the sum of the infinite series 1, $-\frac{1}{2}$, $+\frac{1}{4}$, $-\frac{1}{8}$, etc., of which the ratio is $-\frac{1}{2}$. Ans. $\frac{2}{3}$.

CASE III.

346. **Given any three elements of a geometrical progression, to find either of the other two.**

The formulas established in Arts. 343, 344,

$$l = ar^{n-1}, \qquad (1)$$

$$S = \frac{rl-a}{r-1}, \qquad (2)$$

are *fundamental* ones; and, since they contain the five elements, if any three of these are given, formulas may be deduced for finding the other two. Thus,

The formulas for the first term:

$$a = \frac{l}{r^{n-1}}. \qquad (3)$$

$$a = rl - (r-1)S. \qquad (4)$$

The formulas for the ratio:

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}. \qquad (5)$$

$$r = \frac{S-a}{S-l}. \qquad (6)$$

The formulas for the number of terms, since n enters only as an exponent, would require a knowledge of logarithms for their application.

NOTE. There will be twenty cases, as in arithmetical progression; and when neither of the given formulas contains the required letters, the superfluous letter may be eliminated by combining the two fundamental formulas.

Give the fundamental formulas. The formulas for the first term. For the ratio.

RULE.

Substitute the given quantities in one of the fundamental formulas, or in a formula deduced from them, and reduce the result.

EXAMPLES.

1. Find the first term, when the last term is 405, the ratio 3, and the number of terms 5. Ans. 5.

2. Find the ratio when the first term is 3, the last term 768, and the number of terms 9. Ans. 2.

3. Find the ratio when the first term is 7168, the last term 7, and the sum of the terms 9555. Ans. $\frac{1}{4}$.

4. The last term of a geometrical series is 3072, the ratio 2, and the sum of the terms 6141; required the first term. Ans. 3.

5. The first term of a geometrical series is 2, the ratio 3, and the sum of the terms 6560; required the last term. Ans. 4374.

CASE IV.

347. Given two terms, to insert one or more geometrical means between them.

1. Let a be the first term, r the constant ratio, and n the number of terms; then the terms of the series will be

$$a, \ ar, \ ar^2, \ ar^3, \ ar^4, \ ar^5, \ \ldots\ldots \ ar^{n-1}.$$

Now, by mere inspection of this series, the following properties are obvious: —

If any two terms be taken as extremes, their product is equal to the product of any two means equally distant from them; or,

If the number of terms be odd, the product of the ex-

Repeat the Rule. What properties of a geometrical series are obvious by inspection?

tremes is equal to the square of the middle term; consequently,

A geometrical mean between two quantities is equal to the square root of their product.

2. Again, let a and l be two given terms, and m the number of means to be inserted. Then, let r denote the ratio, and from equation (5), Art. 346, we have

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}.$$

But m represents the number of means; therefore,

$$m + 2 = n,$$

since the number of terms is always two more than the number of means. Hence,

$$\left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{l}{a}\right)^{\frac{1}{m+1}},$$

whence,

$$r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}}.$$

This determines r, and the required means are ar, ar^2, ar^3, etc., and the series,

$$a, \quad ar, \quad ar^2, \quad ar^3, \ldots . ar^m, \quad l.$$

Hence the following

RULE.

Divide the greater of the given terms by the less, and extract the root of the quotient to the degree denoted by the number of means to be inserted plus 1, *for the ratio; and the given first term multiplied by the ratio will give the first of the required means, that multiplied by the ratio will give the second mean, and so on.*

EXAMPLES.

1. Find the geometrical mean between $\frac{1}{3}$ and $\frac{3}{4}$.

Ans. $\frac{1}{2}$.

How is a geometrical mean to be found? Demonstrate the formula for inserting any number of geometrical means between two given terms. Repeat the Rule.

2. Insert two geometrical means between 5 and 320.
Ans. 20, 80.

3. Find the geometrical mean between 30 and 7½.
Ans. 15.

4. Insert three geometrical means between 6 and 486.
Ans. 18, 54, 162.

5. Which is the greater, the arithmetical mean, or the geometrical mean, between 1 and $\frac{1}{9}$; and how much greater? Ans. The arithmetical mean, by $\frac{2}{9}$.

PROBLEMS.

348. The principles of geometrical progression are to be applied, either directly or indirectly, in the solution of the following problems.

1. The first two terms of a series in geometrical progression are ⅓ and 1; what are the next two terms?
Ans. 3 and 9.

2. If the third and fifth terms of a geometrical series are 75 and 300, respectively, what is the sixth term?
Ans. 600.

3. A laborer agrees to labor at the rate of $1 for the first month, $2 for the second, and so on; what is his price for the 10th month? Ans. $512.

4. A person who saved every year half as much again as he saved the previous year, had in seven years saved $102.95. How much did he save the first year?
Ans. $3.20.

5. I wish to discharge a debt in one year by monthly payments in geometrical progression; allowing the first payment to be $1, and the last $2048, what will be the common ratio? Ans. 2.

6. Suppose a body to move 20 miles the first minute,

19 miles the second, $18\frac{1}{20}$ the third, and so on forever; required the utmost distance it can reach.

Ans. 400 miles.

7. I have a rectangular piece of land, 18 rods wide by 288 rods long. What is the side of a square piece containing the same number of square rods?

Ans. 72 rods.

8. If the second term of a geometrical series is 6, and the fourth term 54, what is the first term? Ans. 2.

9. The first and eighth terms of a geometrical progression are 1 and 128, respectively, required the series.

Ans. 1, 2, 4, 8, 16, 32, 64, 128.

10. In the geometrical progression $\frac{x}{y}$, x, xy, required the ratio. Ans. y.

11. A gentleman divided $210 among three servants, the shares being in geometrical progression; and the first had $90 more than the last. How much had each?

Ans. $120, $60, and $30.

If x represent the first term, and y the ratio, then

$$x + xy + xy^2 = 210, \text{ and } x - xy^2 = 90.$$

12. A series of terms are in geometrical progression; the sum of the first two is $1\frac{1}{3}$, and the sum of the next two is 12. Find the series. Ans. $\frac{1}{3}$, 1, 3, 9, etc.

13. The sum of three numbers in geometrical progression is 35, and the mean is to the difference of the extremes as 2 to 3. Required the numbers.

Ans. 5, 10, 20.

14. There are four numbers in geometrical progression, the second of which is less than the fourth by 24, and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers. Ans. 1, 3, 9, 27.

MISCELLANEOUS EXAMPLES.

1. What is the quantity made up of the factors $a x$, $2 b y$, and z? Ans. $2 a b x y z$.

2. Find the value of $2 a^2 - b^2 + x^2$, when $a = 4$, $b = 6$, and $x = -2$. Ans. 0.

3. Find the value of $\frac{2 x^2 - a x}{a^3 - x^3}$, when $a = 2$, and $x = -\frac{1}{2}$.
Ans. $\frac{12}{65}$.

4. Show that $(a^2 + a b + b^2)(a^2 - a b + b^2)$ is equal to $a^4 + a^2 b^2 + b^4$.

5. Factor $48 a^2 b x^3$.
Ans. $2 \times 2 \times 2 \times 2 \times 3 a a b x x x$.

6. What are the factors of $16 a^2 b^2 - 9 x^2$?
Ans. $4 a b + 3 x$, and $4 a b - 3 x$.

7. Simplify $1 - x + \frac{x^2}{1 + x}$. Ans. $\frac{1}{1 + x}$.

8. Simplify $\frac{5 a + 3 x}{3} - (a - x)$. Ans. $\frac{2 a + 6 x}{3}$.

9. One factor of $x^2 + 2 x - 3$ is $x - 1$; find the other.
Ans. $x + 3$.

10. Multiply $\frac{a + 1}{a - 1}$ by $a - 1$.

11. Divide $x - \frac{3 x}{1 + x}$ by $\frac{-2 x}{1 + x}$. Ans. $\frac{2 - x}{2}$.

12. Factor $16 x^2 y^4$ and $28 a x y$, and find the greatest common divisor of the two quantities.

Ans. Factors, $2 \times 2 \times 2 \times 2 x x y y y y$, and $2 \times 2 \times 7 a x y$; greatest common divisor, $4 x y$.

13. A certain garden contained 3 times as many pear-trees as apple-trees. Afterwards 4 of each were cut down, and then there were 4 times as many pear-trees as apple-trees. How many were there of each at first?
Ans. Apple-trees, 12; pear-trees, 36.

14. A man dies, and leaves a widow, two sons, and

three daughters; and in his will he orders that his personal property, amounting to $ 1700, shall be so divided that the three daughters shall have as much as the two sons, and the widow as much as a son and a daughter together. What are their respective shares?

Ans. Each son's share, $ 300; each daughter's, $ 200; the widow's, $ 500.

15. A pump which lifts 2 gallons of water at each stroke, and makes 3 strokes in 2 minutes, is to be replaced by another which can make only 2 strokes in 3 minutes. What must be the discharge of the latter per stroke, to do the same work? Ans. 4½ gallons.

16. A person observes the discharge of a gun at a distance, and hears the report exactly 10½ seconds afterwards. Assuming that light travels at the rate of 192,000 miles, and sound 1090 feet per second, what is the distance between him and the gun?

Ans. 2⅙ miles, nearly.

17. A servant is dispatched on an errand to a town 8 miles off, and walks at the rate of 4 miles an hour; 10 minutes afterwards another is sent to bring him back, walking 4½ miles per hour. How far from the town will the latter overtake the former? Ans. 2 miles.

18. A student has just an hour and a half for exercise. He starts off on a coach which travels 10 miles an hour, and after a time he dismounts, and walks home at the rate of 4 miles an hour. What is the greatest distance he can travel by the coach, so as to keep within his time?

19. An orange-woman bought some oranges, and afterwards forgot the price; but she recollected that she paid for them in shillings and halfpence, that the number of each coin was the same, and that she had as many dozens of oranges as the number of shillings and halfpence taken together. What was the price per dozen? Ans. 6¼*d*.

20. A clock is right at mid-day, Tuesday; but it gains 1 minute per day. When it indicates mid-day on Wednesday, what is the true time?

Ans. $\frac{1440}{1441}$ of a minute to mid-day.

21. A man's principal is \$100,000, some of which is at interest at the rate of 5 per cent, and the rest at 4. His total income is \$4290. Required the portion which produces 5 per cent. Ans. \$29,000.

22. A person bought a lot of cattle for \$180. After reserving 2 of them, he sold the rest for the same sum, \$180. Now he found that he had gained on each, one third more per cent than the cost price of each. How many did he buy? Ans. 12.

23. A person, dying, left his property equally between his two sons. After seven years, the one had quadrupled his money, and the other had lost \$1000, and it was found that the former possessed five times as much as the latter. Required the sum left for each.

24. Gold is $19\frac{1}{4}$ times as heavy as water, and silver $10\frac{1}{2}$ times. A mixed mass weighs 4160 ounces, and displaces 250 ounces of water. What proportion of gold and silver does it contain?

Ans. 3377 ounces of gold; 783 ounces of silver.

25. A boy having worked 12 days, and been idle 5, received 35 cents, the cost of his board having been deducted; but when he worked 16 days, and was idle 7, he received 33 cents. What were his daily wages, and the charge for his board?

Ans. Wages, 61 cents; board, 41 cents.

26. Find a number such, that the sum of two thirds of it and one fourth of it, diminished by 2, shall be equal to eleven twelfths of it plus 3. Ans. $\frac{60}{0}$, or ∞.

27. Find a factor that shall rationalize $a^{\frac{3}{2}} - b^{\frac{5}{2}}$.

28. Simplify $\frac{y - x + \frac{a}{2}}{7\frac{3}{4}}$. Ans. $\frac{4y - 4x + 2a}{31}$.

29. Required the least common multiple of $4(1 - x^2)$, $8(1 - x)$, and $4(1 + x^2)$. Ans. $8(1 - x^4)$.

30. Required the value of .9999, etc., to infinity.
Ans. 1.

31. A can do a piece of work in 20 days, and B and C can perform it in 12 days; but if all three work 6 days, C can finish it in 3 days. In what time would B or C perform it alone?

Ans. B, 60 days; C, 15 days.

32. Required the square factors of 10,000.

Ans. 4, 25, 4, 25.

33. What is the seventh power of $-2a^{\frac{2}{21}}$?
Ans. $-128a^{\frac{2}{3}}$.

34. A pile is one fifth of its whole length in the earth, three sevenths of its length in the water, and 13 feet out of the water. What is the length of the pile?

35. On the 4th of July, 1855, a poor man received from A as many times $4 as A was years old, and a similar gift each July for the seven years following, in the last of which A died, his bounty to the poor man having amounted in all to $1904. What was A's age when he died, and in what year was he born?

Ans. 63 years; A. D. 1799.

36. Two persons, A and B, are traveling on roads which intersect at right angles. B is 540 yards short of the point of crossing when A passes it, and in 2 minutes they are equally distant from that point; also, in 8 minutes more they are again equally distant from it. Required the speed of each?

Ans. A's, 108 yards a minute; B's, 162.

37. There are three numbers in geometrical progression; the sum of the first and second is 10, and the dif-

ference of the second and third is 24. What are the numbers? Ans. 2, 8, and 32.

38. Find two numbers whose difference is 8, and product 105. Ans. 15 and 7, or -7 and -15.

39. Two persons, A and B, set out together from a given point, to travel round the world, a distance of 23661 miles, the one going east and the other west. A goes one mile the first day, two the second, three the third, and so on; B goes 20 miles a day. In how many days will they meet, and how far will each travel?

Ans. They travel 198 days; A goes 19701 miles, and B 3960 miles.

40. Required a fraction, which, if a be added to its numerator, will become b; but if c be added to its denominator, will become d.

$$\text{Ans. } \frac{\dfrac{ad+bcd}{b-d}}{\dfrac{a+cd}{b-d}}.$$

41. Express $x^{-\frac{2}{3}}$ without the use of a negative or a fractional exponent.

42. If two men, working 8 hours a day, can copy a manuscript in 32 days, in how many days can a men, working b hours a day, copy it?

43. Divide the number m into two parts, so that one shall be m times as great as the other.

$$\text{Ans. } \frac{m^2}{m+1} \text{ and } \frac{m}{m+1}.$$

44. A and B go into partnership with a capital of \$416; A's money was in trade 9 months, and B's 6 months; when they shared stock and gain, A received \$228, and B \$252. Required each man's stock.

Ans. A's, \$192; B's, \$224.

45. If the interest of a national debt be \$30,000,000 per annum, and 3 per cent the average rate of interest paid, what reduction in the rate of interest would give

the same relief to taxation as paying $200,000,000 of the debt, allowing the interest paid on the remainder to continue the same? Ans. From 3 to $2\frac{2}{5}$ per cent.

46. Simplify the expression $(2c - 3r)x - (c - 1)x - (c - 2r)x - x$. Ans. $-rx$.

47. How much does $\frac{1}{a - b}$ differ from $\frac{1}{a} - \frac{1}{b}$?

48. How does $\frac{a - b}{x}$ compare with $\frac{a}{x} - \frac{b}{x}$?

49. Divide $a - b$ by $\sqrt{a} - \sqrt{b}$.

50. A countryman being employed by a poulterer to drive a flock of geese and turkeys to London, in order to distinguish his own from any he might meet on the road, pulled three feathers out of the tail of each turkey, and one out of the tail of each goose; and, upon counting them, found that the number of turkey's feathers exceeded twice those of the geese by 15. Having bought 10 geese and sold 15 turkeys by the way, he was surprised to find, as he drove them into the poulterer's yard, that the number of geese exceeded the number of turkeys in the proportion of 7 to 3. Required the number of each at first. Ans. 45 turkeys; 60 geese.

51. In the composition of a certain quantity of gunpowder, two thirds of the whole, plus 10 pounds, was nitre; one sixth of the whole, less $4\frac{1}{2}$ pounds, was sulphur; and the charcoal was one seventh of the nitre, less 2 pounds. How many pounds of gunpowder were there? Ans. 69.

52. Iron worth $10 in its raw state is manufactured half into knife-blades and half into razors, and is then worth $444. But if one third of it had been made into knife-blades, and the rest into razors, the produce would have been worth $30 more than in the former case. How much is one dollar's worth of the original material increased in value by these respective manufactures?

Ans. $52.40 in razors; $34.40 in knife-blades.

53. A clergyman, who had a charity of $ 110 to distribute among a certain number of old men and widows, found that, if he gave $ 3 to each, he would be one dollar out of pocket; but if he gave each of the men $ $2\frac{1}{6}$, and each of the widows $ 3.50, he would have 50 cents to spare. How many were there of each?

Ans. 15 men; 22 women.

54. The year of our Lord in which the "change of style" took place possesses the following properties: the first digit being 1 for thousands, the second is the sum of the third and fourth, the third is the *third* part of the sum of all four, and the fourth is the *fourth* part of the sum of the first two. Determine the year.

Ans. A.D. 1752.

55. Seven horses and four cows consume a stack of hay in 10 days, and two horses can eat it alone in 40 days; in how many days will one cow be able to eat it?

Ans. 320 days.

56. A farmer bought a certain number of sheep for $ 94; he lost 7 of them, and sold one fourth of the remainder at *prime cost* for $ 20. How many did he buy?

Ans. 47 sheep.

57. A person had a certain number of coins, of equal size, which he tried to arrange in the form of a square, placing them as close together as possible on the table. At the first trial he had 130 coins over; but when he enlarged the side of the square by 3, he had only 31 over. How many had he? Ans. 355 coins.

58. A certain wagon has a mechanical contrivance which marks the difference of the number of revolutions of the fore and hind wheels in any journey. The rim of each fore wheel is $5\frac{1}{4}$ feet, and of each hind wheel $7\frac{1}{8}$ feet; find the distance traveled when the fore wheel has made exactly 2000 revolutions more than the hind wheel. Ans. $7\frac{1}{2}$ miles 100 yards.

59. There is a wall containing 5400 cubic feet. The height is 5 times the thickness, and the length 8 times its height. What are the dimensions?

Ans. 3 feet thick, 15 feet high, and 120 feet long.

60. In a fleet of transports, the square root of half the number of ships expressed the number bound for the Gulf of Mexico, $\frac{7}{8}$ of the fleet were for the Pacific coast, and the remaining 8 for coast defence. Required the whole number. Ans. 128 ships.

61. A regiment of 594 men is to be raised from three towns, A, B, and C. The quotas of A and B are in the proportion of three to five; and of B and C, in the proportion of eight to seven. Required the number raised by each. Ans. 144 by A, 240 by B, and 210 by C.

62. A farm consisting of two kinds of land is let at an annual rent of $390, the pasture being valued at $1.50 per acre, and the arable land at $3. Now the number of acres of arable land is to half the excess of the arable land above the pasture as 5 to 1. Required the quantity of each.

Ans. 60 acres pasture; 100 acres arable land.

63. A merchant sold to one person 9 chests of tea and 7 bags of coffee for $300; and to another, at the same prices, 6 chests of tea and 13 bags of coffee for the same sum. What was the price of each?

Ans. Tea, $24 a chest; coffee, $12 a bag.

64. How far does a person travel in gathering up 200 apples placed in a straight line, at intervals of 2 feet from each other, supposing that he brings each apple singly and deposits it in a basket, which is in the same line produced, 20 yards from the nearest apple, and that he starts from the basket? Ans. $19\frac{41}{66}$ miles.

65. Required the arithmetical mean between $1+x$ and $1-x$.

66. A farmer sowed a peck of wheat, and used the whole produce for seed the following year, the produce of this second year again for seed the third year, and the produce of this again for the fourth year. He then sells his stock after harvest, and finds that he has $12656\frac{1}{4}$ bushels to dispose of. Supposing the increase to have been always in the same proportion to the seed sown, what was the annual increase? Ans. 15 times.

67. Find the sum of 4 terms of a series in geometrical progression, of which the first term is $\frac{16}{27}$, and the fourth is 2. Ans. $4\frac{22}{27}$.

68. A gardener undertook to plant a number of trees at equal distances apart, and in the form of a square. In the first attempt, when he had finished his square, he had 11 trees left. He then added one to each row, as far as they would go, and found that he wanted 24 trees more to complete his square. How many trees were there? Ans. 300.

69. At what times between 1 o'clock and 2 o'clock is there exactly one minute space between the two hands of a clock? Ans. $4\frac{4}{11}$ or $6\frac{6}{11}$ minutes past 1.

70. The difference of the squares of two consecutive numbers is 15; what are the numbers? Ans. 7 and 8.

71. A certain number is formed by the product of three consecutive numbers; and if it be divided by each of them in turn, the sum of the quotients is 47. What is the number? Ans. 60.

72. A gentleman is 39 years old, and his son is 17. In how many years will the father be three times as old as his son?

73. A boat's crew row $3\frac{1}{2}$ miles down a river and back again in 1 hour 40 minutes. Supposing the river to have a current of 2 miles per hour, find the rate at which the crew would row in still water. Ans. 5 miles per hour.

74. Find two numbers whose sum is 9 times their difference, and whose product is equal to 12 times their quotient, together with the greater number.

Ans. 5 and 4.

75. Two travelers, A and B, set out at the same time from two places, M and N, respectively, and travel so as to meet. When they meet, it is found that A has traveled 30 miles more than B; that A will reach N in 4 days, and B will reach M in 9 days, after they meet. Find the distance between M and N. Ans. 150 miles.

76. Find that whole number whose square added to its cube is nine times the next higher whole number.

Ans. 3.

77. A ship sails with a supply of biscuit for 60 days, at a daily allowance of 1 lb. a head. After being at sea 20 days, she encounters a storm, in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, when it is found that each man's allowance must be reduced to five sevenths of a pound. Find the original number of the crew. Ans. 40 men.

78. One cask contains 12 gallons of wine and 18 gallons of water, and another cask contains 9 gallons of wine and 3 gallons of water; how many gallons must be drawn from each cask so as to produce by their mixture 7 gallons of wine and 7 gallons of water?

Ans. 10 gal. from one, 4 gal. from the other.

79. A vessel can be filled with water by two pipes; by one of these pipes alone the vessel would be filled 2 hours sooner than by the other; also the vessel can be filled by both together in $1\frac{7}{8}$ hours. Find the time which each pipe alone would take to fill the vessel.

Ans. First, 3 hours; second, 5 hours.

80. The sum of the reciprocals of three numbers is 9: the sum of two times the reciprocal of the first and three

times the reciprocal of the second is 13; and the sum of eight times the first and three times the second is 5. What are the numbers?

Ans. First, $\frac{1}{2}$, or $\frac{5}{26}$; second, $\frac{1}{3}$, or $\frac{15}{13}$; third, $\frac{1}{4}$, or $\frac{15}{44}$.

81. A person leaves $12,670 to be divided among his five children and three brothers, so that, after the legacy duty has been paid, each child's share shall be twice as great as each brother's. If the legacy duty on a child's share were one per cent, and on a brother's share three per cent, find what amounts they would respectively receive.

Ans. Each child, $1920.60; each brother, $960.30.

82. There is a number consisting of two digits; the number is equal to three times the sum of its digits, and if it be multiplied by three, the result will be equal to the square of the sum of its digits. What is the number? Ans. 27.

83. A sets out from a certain place and travels one mile the first day, two miles the second day, three the third, and so on. B sets out from the same place five days after A, and travels the same road, at the rate of 12 miles a day. How far will A travel before the two will be together? Ans. 36 miles, or 120 miles.

84. From 256 gallons of vinegar a certain number are drawn and replaced with water; this is done a second, a third, and a fourth time, and 81 gallons of vinegar are then left. How much was drawn out each time?

Ans. 64 gallons.

85. How many terms of the natural numbers, commencing with 4, give a sum of 5350? Ans. 100.

86. There are four numbers, the first three of which are in geometrical progression, and the last three in arithmetical progression; the sum of the first and last is 14,

and that of the second and third is 12. What are the numbers? Ans. 2, 4, 8, 12; or, $\frac{25}{2}$, $\frac{15}{2}$, $\frac{9}{2}$, $\frac{3}{2}$.

87. Three numbers, whose sum is 15, are in arithmetical progression; but if 1, 4, and 19 be added to them, respectively, they are in geometrical progression. Determine the numbers. Ans. 2, 5, and 8.

88. Two men, A and B, bought a farm of 200 acres, for which they paid $ 200 each. On dividing the land, A says to B, "If you will let me have my part in the situation which I shall choose, you shall have so much more land than I that mine shall cost 75 cents per acre more than yours." B accepted the proposal. How much land did each have, and what was the price of each per acre?

Ans. A, 81.867 A., at $ 2.443; B, 118.133 A., at $ 1.693.

89. A and B engaged to reap a field for 90 shillings. A could reap it in 9 days, and they promised to complete it in 5 days. They found, however, that they were obliged to call in C, an inferior workman, to assist them the last two days, in consequence of which B received 3 s. 9 d. less than he otherwise would have done. In what time could B and C reap the field?

Ans. B could reap it in 15 days; C, in 18 days.

90. A certain number of workmen can move a heap of stones in 8 hours from one place to another. If there had been 8 more workmen, and each workman had carried 5 lbs. less at a time, the whole work would have been completed in 7 hours. If, however, there had been 8 fewer workmen, and each had carried 11 lbs. more at a time, the work would have occupied 9 hours. Find the number of workmen, and the weight which each carried at a time.

Ans. 36 workmen, each carrying 77 lbs. at a time; or, 28 workmen, each carrying 45 lbs. at a time.

THE END

HARVARD COLLEGE
LIBRARY

THE ESSEX INSTITUTE
TEXT-BOOK COLLECTION

GIFT OF
GEORGE ARTHUR PLIMPTON
OF NEW YORK

JANUARY 25, 1924